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Lesson - 1

Limits

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Structure

1.1 Learning Objectives

1. Understanding the concept of limit informally using graphs and numerical calculations.

2. Understanding the concept of left-hand limit and right-hand limit.

3. Understanding $\epsilon - \delta$ definition of limit.

4. Understanding infinite limits and limits at infinity.

5. Understanding vertical and horizontal asymptotes of a curve.

1.2 Introduction

The concept of *limit* is fundamental to the study of Calculus. In the subsequent lessons, we will see how the definition of *limit* is used in the other important concepts like continuity, differentiability, curve tracing, mean value theorems, and intermediate forms. The concept of *limit* is very much similar to the notion of closeness. It basically talks about how a function behaves when its independent variable is close to a particular value. First, we will informally introduce the concept of *limit* using graphs and numerical calculations. Later we will define it more formally.

1.3 Informal approach to Limit

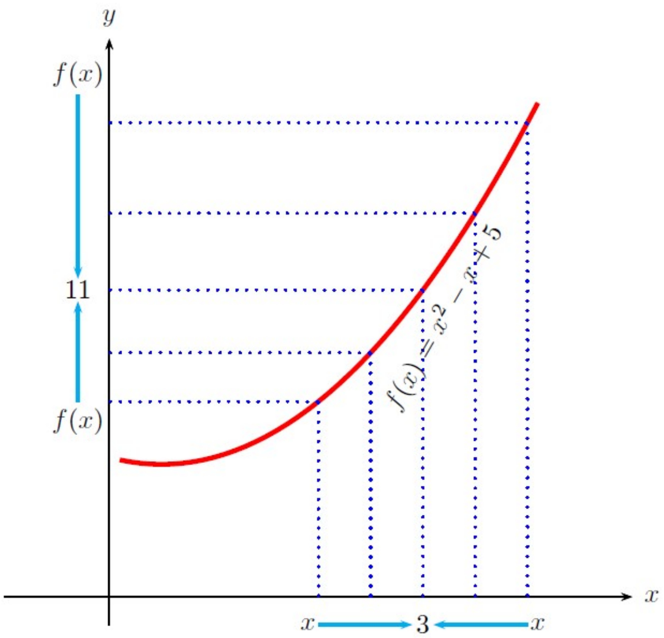


Figure 1.1: Graph of the function $f(x) = x^2 - x + 5$.

$x < 3$		$x > 3$	
x	$f(x)$	x	$f(x)$
2	7	4	17
2.5	8.75	3.5	13.75
2.9	10.51	3.1	11.51
2.99	10.95	3.01	11.05
2.999	10.99	3.001	11.005

Table 1.1: Values of function $f(x) = x^2 - x + 5$ for x close to 3.

To get an informal idea of limit, consider the graph of the function $f(x) = x^2 - x + 5$ shown in the Figure 1.1. We can see that as x approaches closer to 3 (not equal to 3) from either left side of 3 ($x < 3$) or right side of 3 ($x > 3$), the function values approach closer to 11. The same is also evident from the Table 1.1. We express this fact as - "limit of the function $f(x) = x^2 - x + 5$ is 11, as x tends to 3" and denotes it by writing:

$$f(x) = x^2 - x + 5 \to 11 \quad \text{when} \quad x \to 3$$

more formally

$$\lim_{x\to 3}(x^2 - x + 5) = 11$$

Based on the above discussion, let us informally define the *Limit* of a function at a given point.

Definition 1.1. (Informal Definition of Limit)
If the function $f(x)$ takes values close to a finite real number L, by taking values of x sufficiently close to a (but not equal to a), then we say that limit of$f(x)$ is L, as x tends (approaches) to a. We write it as:

$$f(x) \to L \quad when \quad x \to a$$

or

$$\lim_{x\to a} f(x) = L$$

Remark. One must note that in the above definition and discussion, we are not considering the function value at $x = a$. We only consider the function values close to $x = a$ from both left ($x < a$) and right ($x > a$) of a.

Example 1.1. Examine the limit of the following function by taking the values of x near 3 and plotting the graph of the function.

$$f(x) = \begin{cases} x^2 - x + 5 & x \neq 3 \\ 0 & x = 3 \end{cases}$$

Solution. It is evident from the Table 1.1 and the graph of the function in Figure 1.2, that $f(x)$ takes values close to 11 as x takes values close to $x = 3$ from both (left as well as right) sides. Hence by using the definition 1.1, we can conclude that

$$\lim_{x\to 3} f(x) = 11$$

Remark. One must note that in the above example though $f(3) = 0$, but the limit definition does not consider the function value at $x = 3$.

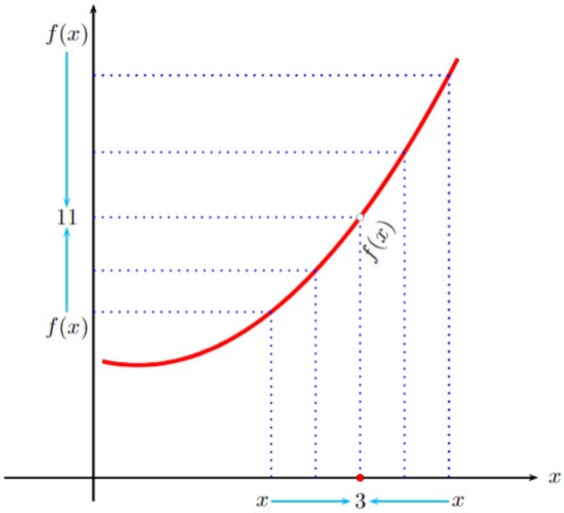


Figure 1.2: Graph of the given function $f(x)$.

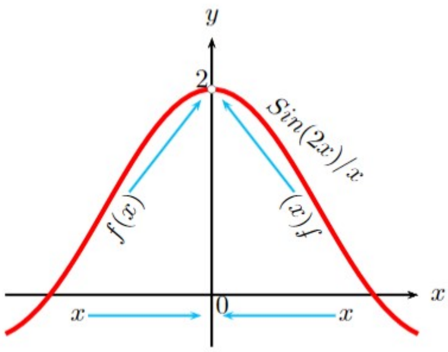


Figure 1.3: Graph of the function $f(x) = \sin(2x)/x$.

x	$sin(2x)/x$
$\pm$ 1	0.909297
$\pm$ 0.5	1.68294
$\pm$ 0.1	1.98669
$\pm$ 0.01	1.99987

Table 1.2: Values of the function $f(x) = sin(2x)/x$ for x close to 0.

Example 1.2. Use definition 1.1, to show that

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = 2$$

Solution. From the graph of the function shown in Figure 1.3 and Table 1.2, one can observe that the function takes values close to 2, as x takes values close to 0 from both sides. Hence by using the definition 1.1, we can conclude that

$$\lim_{x\to 0} \frac{\sin(2x)}{x} = 2$$

Remark. Here the function $f(x) = \frac{Sin(2x)}{x}$ is not defined at $x = 0$.

We can not always rely on numerical values of $f(x)$ to conclude about the existence of the limit of $f(x)$ at a point. For example, consider the numerical values of the function $f(x) = sin(\pi/x)$ for the values of x close to zero, as shown in Table 1.3. It gives an impression as if the function has a limit 0 at the point $x = 0$, but this is not true. From the graph of the function shown in Figure 1.4, one can observe that the function oscillates between -1 and 1 with increasing frequency as x approaches towards zero. So we conclude that the limit does not exist at $x = 0$.

x	π/x	$sin(\pi/x)$
$\pm$ 1	$\pm\pi$	0
$\pm$ 0.1	$\pm 10\pi$	0
$\pm$ 0.01	$\pm 100\pi$	0
$\pm$ 0.001	$\pm 1000\pi$	0
$\pm$ 0.0001	$\pm 10000\pi$	0

Table 1.3: Function $f(x) = sin(\pi/x)$ values close to 0.

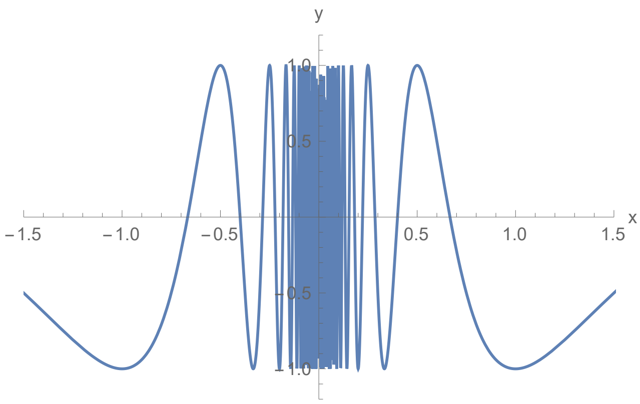


Figure 1.4: Graph of $sin(\pi/x)$

1.4 One-sided Limits

The Definition 1.1 for the limit L of a function $f(x)$ at $x = a$, considers the values of $f(x)$ approaching to L from above ($f(x) > L$) and from below ($f(f) < L$) as x approaches to a from left ($x < a$) and from right ($x > a$). Therefore L is called two sided limit of

$f(x)$ as x tends to a. Some functions behave differently on the two sides of point a. For example, note the graph of the function $|x|/x$ shown in the Figure 1.5. One can see the function approaches to 1 if x approaches to zero from the right and it approaches to -1 if x approaches zero from the left. In view of this observation, we introduce one-sided limits namely - the left-hand limit and right-hand limit of a function at a point.

Definition 1.2. Informal definition of one-sided limits

(**Right-hand Limit:**) Let a function $f(x)$ be defined for all $x > a$. If $f(x)$ takes values close to L, by taking values of x sufficiently close to a from the right side, that is, for $x > a$, then we write

$$f(x) \to L \quad when \quad x \to a^+$$

or

$$\lim_{x \to a^+} f(x) = L$$

and L is called the right-hand limit of $f(x)$ at a.
(Here $x \to a^+$ means x approaches to a from right, that is for $x > a$.)

(**Left-hand Limit:**) Let a function $f(x)$ be defined for all $x < a$. If $f(x)$ takes values close to L, by taking values of x sufficiently close to a from the left side, that is, for $x < a$, then we write

$$f(x) \to L \quad when \quad x \to a^-$$

or

$$\lim_{x \to a^-} f(x) = L$$

and L is called the left-hand limit of $f(x)$ at a.
(Here $x \to a^-$ means x approaches to a from left, that is for $x < a$.)

Example 1.3. If $f(x) = |x|/x$, show that:

$$\lim_{x->0^+} f(x) = 1 \quad \text{and} \lim_{x->0^-} f(x) = -1$$

.

Proof. We know $|x| = x$ if $x > 0$ and $|x| = -x$ if $x < 0$, this implies

$$f(x) = \begin{cases} x/x = 1 & \text{if} \quad x > 0 \\ -x/x = -1 & \text{if} \quad x < 0 \end{cases} \tag{1.1}$$

Thus as x approaches to 0 from left (that is $x < 0$), $f(x)$ approaches to -1 and as x approaches to 0 from right (that is $x > 0$), $f(x)$ approaches to 1. So $\lim_{x->0^+} f(x) = 1$ and $\lim_{x->0^-} f(x) = -1$.

One can also conclude the same with the help of the graph of the function shown in the Figure 1.5. □

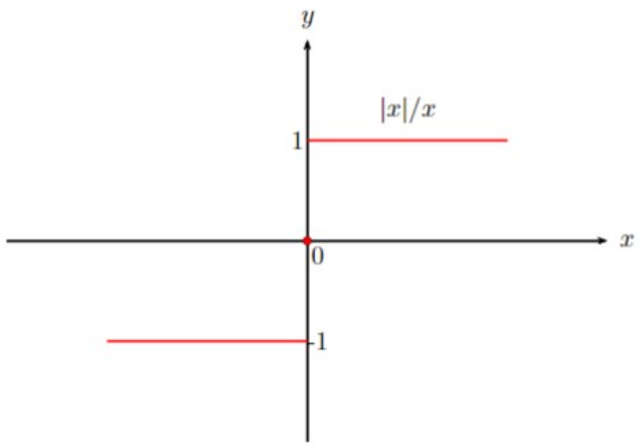


Figure 1.5: Graph of $|x|/x$

Remark. From the Definitions 1.1 and 1.2, one can conclude that

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

where L is the finite real number. We have the following theorem.

Theorem 1.1. ***(Necessary and Sufficient Condition)***
Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, then limit $\lim_{x\to a} f(x)$ exists if and only if both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exists and they are equal.

Theorem 1.2. ***(Non-existence of Limit)*** *The limit $\lim_{x\to a} f(x)$ does not exist if:*

(i) either of the one-sided limit does not exist, that is $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ does not exist.

(ii) both one sided-limits exist but are unequal, that is $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$.

Since in the example 1.3, we have:

$$\lim_{x\to 0^+} f(x) = 1 \neq -1 = \lim_{x\to 0^-} f(x)$$

Therefore, we can conclude that $\lim_{x\to 0} f(x)$ does not exist in this case.

1.5 $\epsilon - \delta$ Definition of Limit (Formal Definition)

The graphical and numerical technique to calculate the limit of a function presented in the last sections is insufficient to establish various results (theorems) involving limits. We need a precise definition that can establish limit of a function without any doubt, and that can be used to establish important results about limits. As seen in definition 1.1, the notion

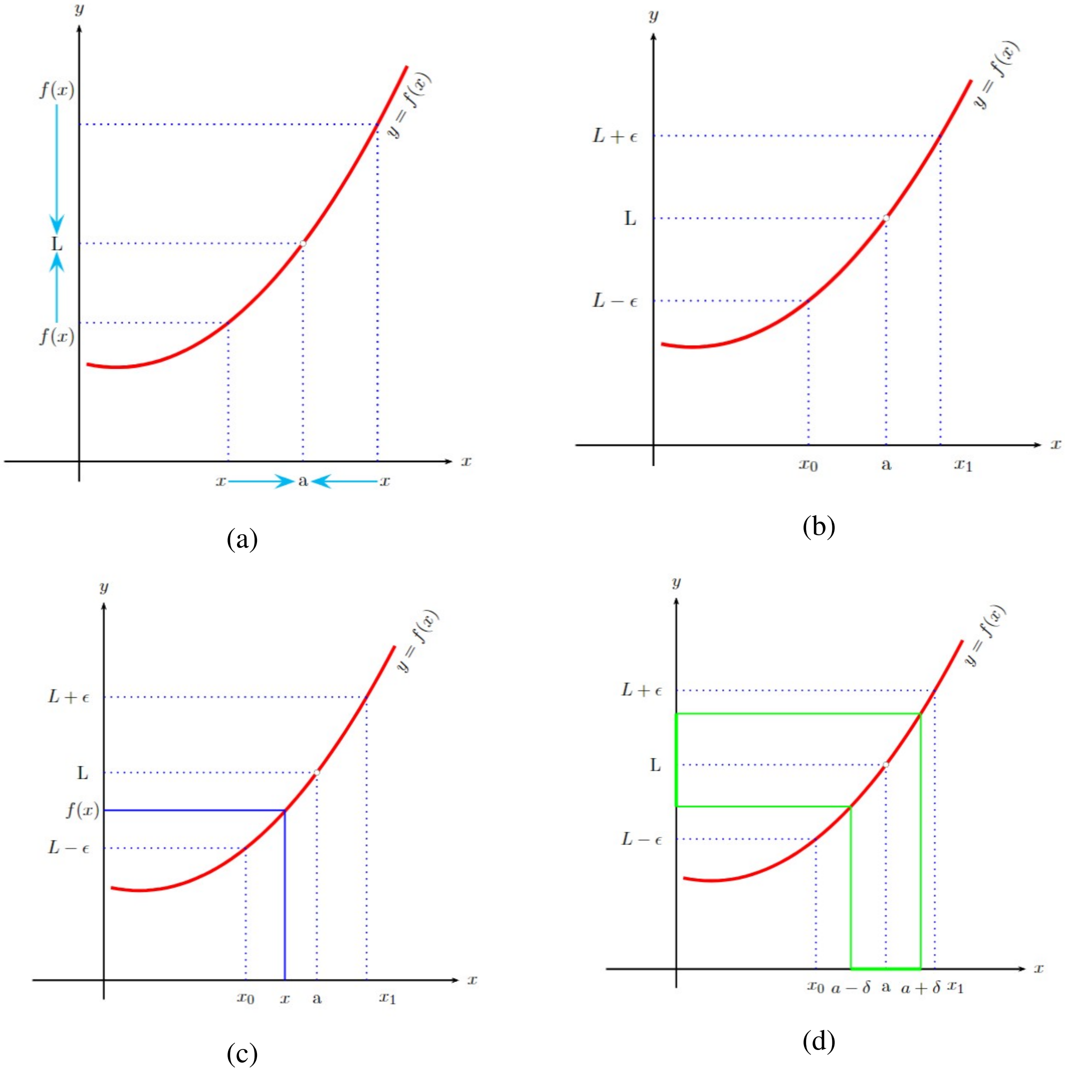


Figure 1.6: Understanding $\epsilon - \delta$ definition of limit.

$\lim\limits_{x \to a} f(x) = L$ means that the values of the function $f(x)$ can be made close (as close as we want) to L, by taking values of x sufficiently close to a (not necessarily at a). We need to define the notions of 'as close as we want' and 'sufficiently close to a' mathematically.

Consider the function $f(x)$ shown in the Figure 1.6(a), which converges to L as x approaches to a. We have shown a blank space in the graph at $x = a$ to emphasize that f need not be defined at $x = a$. Now lets try to capture the notion of 'making values of f as close as we want to L' by saying that the values of $f(x)$ lies in between $L - \epsilon$ and $L + \epsilon$ (that is $L - \epsilon < f(x) < L + \epsilon$), where we can choose $\epsilon > 0$ as much small as we want. Now the challenge is to determine how much close we need to go $x = a$ to ensure $L - \epsilon < f(x) < L + \epsilon$. To answer this, let us do the following.

Draw horizontal lines from the points $L - \epsilon$ and $L + \epsilon$ on $y - axis$ till they intersect the curve $y = f(x)$ as shown in Figure 1.6(b). Now draw vertical lines from these intersection points on the curve till the $x - axis$. Let these vertical lines meets at the points x_0 and

x_1 respectively on $x - axis$. This impels that whenever x lies in the open interval (x_0, x_1) containing a, $f(x)$ lies in the open interval $(L - \epsilon, L + \epsilon)$, containing L, as shown in Figure 1.6(c).

Notice that the open interval (x_0, x_1), extends more in left (towards x_0) than right (towards x_1). To make the formal definition of limit that can be used to derive many important results regarding limits, it is useful to have an equidistant interval centered at a. For this purpose lets choose a number $\delta < Min(a - x_0, x_1 - a)$, then one can see that whenever x lies in the open interval $(a - \delta, a + \delta)$, $f(x)$ lies in the open interval $(L - \epsilon, L + \epsilon)$, as shown in Figure 1.6(d).

The fact that $f(x)$ lies in the open interval $(L-\epsilon, L+\epsilon)$ can be captured as $|f(x)-L| < \epsilon$ and the fact x lies in the open interval $(a-\delta, a+\delta), x \neq a$ can be captured as $0 < |x-a| < \delta$. With this discussion, let's define the limit of a function in a more precise mathematical way.

Definition 1.3. (Formal definition of limit)

Let f be a function defined on an open interval containing a, but not necessarily at a. We say that $\lim\limits_{x \to a} f(x) = L$, if given any $\epsilon > 0$, there exists a $\delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

.

Remark. One must note that the value of δ is not unique. In particular, if $\delta = \delta_0$ satisfies the conditions in the above definition, then any other value $\delta < \delta_0$ will also satisfy the conditions.

Example 1.4. By using $\epsilon - \delta$ definition, prove that $\lim\limits_{x \to a} c = c$, where c is any real number.

Solution. Here we have $f(x) = c, L = c$. We need to show that given any $\epsilon > 0$, there exists a $\delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

that is $|f(x) - c| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$.

Here $|f(x) - c| = |c - c| = 0 < \epsilon$, is always true.

Hence by definition 1.3, $\lim\limits_{x \to a} c = c$.

Example 1.5. By using $\epsilon - \delta$ definition, prove that $\lim\limits_{x \to a} x = a$, where a is any real number.

Solution. Here we have $f(x) = x, L = a$. We need to show that given any $\epsilon > 0$, there exists a $\delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

that is $|f(x) - a| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$.

Here $|f(x) - a| = |x - a| < \epsilon$, whenever $|x - a| < \delta(= \epsilon)$.

Hence by definition 1.3, $\lim\limits_{x \to a} x = a$.

Example 1.6. By using $\epsilon - \delta$ definition, prove that $\lim\limits_{x \to 1}(2x + 3) = 5$.

Solution. Here we have $f(x) = 2x + 3, L = 5, a = 1$. We need to show that given any $\epsilon > 0$, there exists a $\delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

that is $|f(x) - 5| < \epsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta$.

$$\begin{aligned} \text{Now} \quad & |f(x) - L| < \epsilon \\ \text{whenever} \quad & |(2x + 3) - 5| < \epsilon \\ \text{or} \quad & |2x - 2| < \epsilon \\ \text{or} \quad & 2|x - 1| < \epsilon \\ \text{or} \quad & |x - 1| < \epsilon/2 \end{aligned}$$

Therefore for $\epsilon > 0$, there exists $\delta = \epsilon/2$, such that $|f(x) - 5| < \epsilon$ whenever $|x - 1| < \delta (= \epsilon/2)$.

Hence by definition 1.3, $\lim_{x \to 1} (2x + 3) = 5$.

Note. Note that in the above example, the inequality $|f(x) - L| < \epsilon$, is true for $|x - a| < \delta$ (that is, for $x = a$ also), although we need it for $0 < |x - a| < \delta$ only.

To further understand, how the choice of ϵ affects the resulted δ in the above example one can have a look at Figure 1.7. One can see how for different values of $\epsilon = 1, 0.5, 0.25, 0.1$ and corresponding values of $\delta = 0.5, 0.25, 0.125, 0.05$ respectively ensures that the function values lies in the interval $(5 - \epsilon, 5 + \epsilon)$.

In-text Exercise 1.1. By using $\epsilon - \delta$ definition, prove that $\lim_{x \to 2} (4x - 3) = 5$.

Example 1.7. By using $\epsilon - \delta$ definition, prove that $\lim_{x \to 1} |x - 1| = 0$.

Solution. Here we have $f(x) = |x - 1|, L = 0, a = 1$. We need to show that given any $\epsilon > 0, \exists \quad \delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < \quad |x - a| < \delta$$

that is, $|f(x) - 0| < \epsilon \quad \text{whenever} \quad 0 < |x - 1| < \delta$.

$$\begin{aligned} \text{Now} \quad & |f(x) - L| < \epsilon \\ \text{whenever} \quad & ||x - 1| - 0| < \epsilon \\ \text{or} \quad & |x - 1| < \epsilon \end{aligned}$$

Therefore for $\epsilon > 0$, there exists $\delta = \epsilon$, such that $|f(x) - 0| < \epsilon$ whenever $|x - 1| < \delta (= \epsilon)$.

Hence by definition 1.3, $\lim_{x \to 1} |x - 1| = 0$.

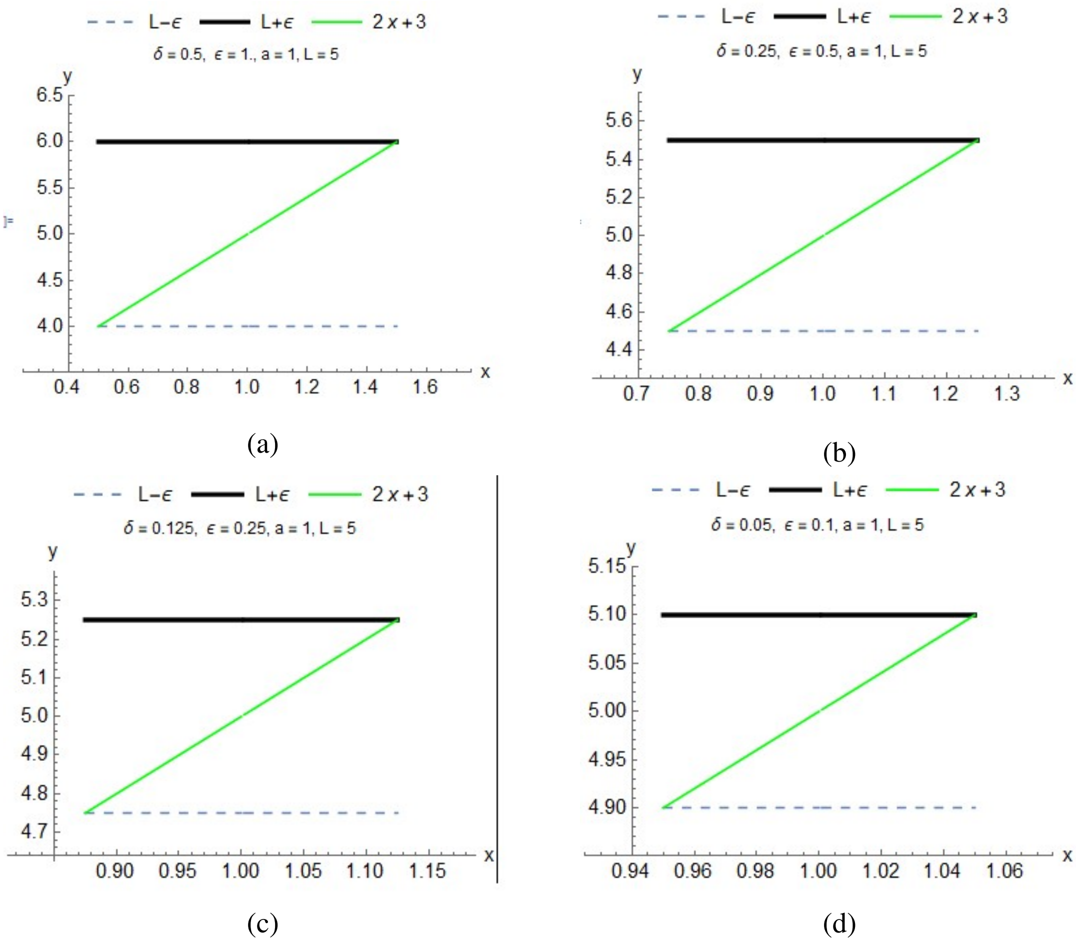


Figure 1.7: Graphical representation of $f(x) \in (5-\epsilon, 5+\epsilon)$ for corresponding $x \in (1-\delta, 1+\delta)$

Example 1.8. Prove that $\lim_{x\to 0} Sin(x) = 0$, by using $\epsilon - \delta$ definition.

Solution. Here we have $f(x) = Sin(x), L = 0, a = 0$. We need to show that given any $\epsilon > 0, \exists \quad \delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < \quad |x - a| < \delta$$

that is $|f(x) - 0| < \epsilon \quad$ whenever $\quad 0 < |x - 0| < \delta$.

$$\begin{aligned} \text{Now} \quad & |f(x) - L| < \epsilon \\ \text{whenever} \quad & |Sin(x) - 0| < \epsilon \\ \text{or} \quad & |Sin(x)| < \epsilon \end{aligned}$$

As $|Sin(x)| \leq |x|$ for all $x \in \mathbb{R}$, so if we take $\delta = \epsilon$, then $|Sin(x)| < \epsilon$ whenever $|x - 0| < \delta(= \epsilon)$.

Hence by definition 1.3, $\lim_{x \to 0} Sin(x) = 0$.

Example 1.9. Prove that $\lim_{x \to 0} xSin(1/x) = 0$, by using $\epsilon - \delta$ definition.

Solution. Here we have $f(x) = xSin(1/x), L = 0, a = 0$. We need to show that given any $\epsilon > 0, \exists \quad \delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < \quad |x - a| < \delta$$

that is, $|f(x) - 0| < \epsilon \quad$ whenever $\quad 0 < |x - 0| < \delta$.

$$\begin{aligned} \text{Now} \quad |f(x) - L| &= |xSin(1/x)| \\ &= |x||Sin(1/x)| \\ &\leq |x|, \quad \text{as} \quad |Sin(1/x)| \leq 1 \quad \text{for all} \quad x \in \mathbb{R} \end{aligned}$$

If we take $\delta = \epsilon$, then $|xSin(1/x)| < \epsilon$ whenever $|x - 0| < \delta(= \epsilon)$.
Hence by definition 1.3, $\lim_{x \to 0} xSin(1/x) = 0$.

Example 1.10. Prove that $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6$, by using $\epsilon - \delta$ definition.

Solution. Here we have $f(x) = \frac{x^2-9}{x-3}, L = 6, a = 3$. We need to show that given any $\epsilon > 0, \exists \quad \delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < \quad |x - a| < \delta$$

that is $|f(x) - 6| < \epsilon \quad$ whenever $\quad 0 < |x - 3| < \delta$.

$$\begin{gathered} \text{Now} \quad |f(x) - L| < \epsilon \\ \text{whenever} \quad |\frac{x^2 - 9}{x - 3} - 6| < \epsilon \\ \text{or} \quad |\frac{(x-3)(x+3)}{x - 3} - 6| < \epsilon \\ \text{or} \quad |x + 3 - 6| < \epsilon \\ \text{or} \quad |x - 3| < \epsilon \end{gathered}$$

Therefore for $\epsilon > 0$, there exists $\delta = \epsilon$, such that $|f(x) - 6| < \epsilon$ whenever $|x - 3| < \delta(= \epsilon)$.

Hence by definition 1.3, $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6$.

Example 1.11. Find $\lim_{x \to 0^-} \lfloor x \rfloor$ and $\lim_{x \to 0^+} \lfloor x \rfloor$, where $[x]$ denotes the greatest integer function. Does $\lim_{x \to 0} \lfloor x \rfloor$ exist?

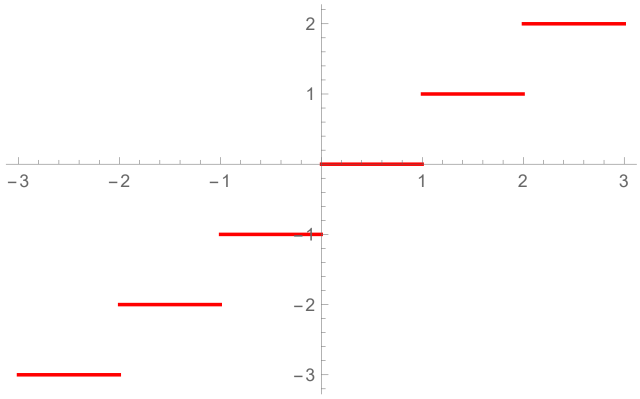


Figure 1.8: Graph of $\lfloor x \rfloor$

Solution. By definition $\lfloor x \rfloor = n$ if $n \leq x < n + 1$, where $n \in \mathbb{Z}$. See Figure 1.8 for a graphical representation of $\lfloor x \rfloor$

In particular $\lfloor x \rfloor = -1$ if $-1 \leq x < 0$ and $\lfloor x \rfloor = 0$ if $0 \leq x < 1$.

As x approaches to zero from left ($x < 0$), the function $\lfloor x \rfloor$ takes value equal to -1, so by Definition 1.2, we ahve $\lim\limits_{x \to 0^-} \lfloor x \rfloor = -1$.

Similarly, as x approaches to zero from right ($x > 0$), the function $\lfloor x \rfloor$ takes value equal to 0, so by Definition 1.2, we have $\lim\limits_{x \to 0^+} \lfloor x \rfloor = 0$.

Since $\lim\limits_{x \to 0^-} \lfloor x \rfloor \neq \lim\limits_{x \to 0^+} \lfloor x \rfloor$, $\lim\limits_{x \to 0} \lfloor x \rfloor$ does not exist.

1.6 Infinite Limits

Sometimes a function increases or decreases at a point without bound. For example consider the function $f(x) = 1/x^2$ shown in Figure 1.9(a). The function values are positive and keep increasing without bounds as x approaches to 0 from both sides. The same can be observed from the Table 1.4. We express this fact by writing:

$$\lim_{x \to 0} 1/x^2 = \infty$$

.

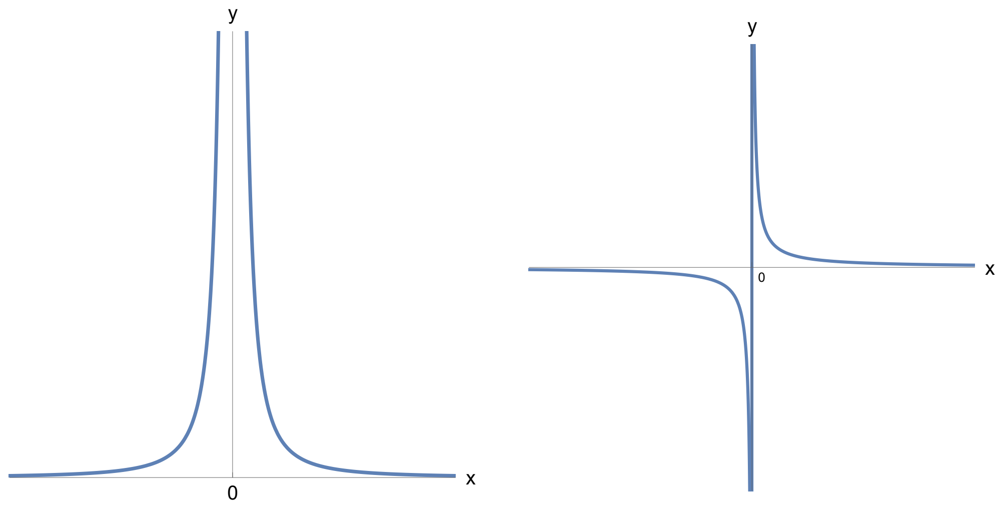
x	$1/x^2$	$1/x$
$\pm$ 1	1	$\pm$1
$\pm$ 0.1	100	$\pm$10
$\pm$ 0.01	10000	$\pm$100
$\pm$ 0.001	1000000	$\pm$1000
$\pm$ 0.0001	100000000	$\pm$10000

Table 1.4: Values of $f(x) = 1/x^2$, $f(x) = 1/x$ for the values of x close to 0.

Now consider the graph of the function $f(x) = 1/x$ shown in the Figure 1.9(b). The function remains positive and keeps increasing without bound if x approaches to 0 from the right ($x > 0$). Also the function remains negative and keeps decreasing without bound if x approaches to 0 from left ($x < 0$). The same can be observed from the Table 1.4. We denote these facts by writing:

$$\lim_{x\to 0^+} 1/x = \infty \quad \text{and} \lim_{x\to 0^-} 1/x == -\infty$$

.



(a) $f(x) = 1/x^2$

(b) $f(x) = 1/x$

Figure 1.9: Infinite Limits at $x = 0$

Definition 1.4. **(Informal view of Infinite Limit)**

Suppose a function $f(x)$ is defined for x near a point a, we say that $f(x)$ tends to ∞ as $x \to a$ and denote it as $\lim_{x\to a} f(x) = \infty$ if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \infty$.

Similarly, we say that $f(x)$ tends to $-\infty$ as $x \to a$ and denote it as $\lim_{x\to a} f(x) = -\infty$ if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = -\infty$.

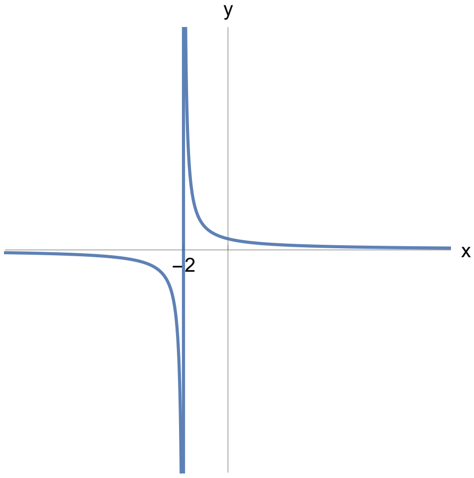
Example 1.12. Use graph of the function $f(x) = 1/(x + 2)$ and $f(x) = 1/(x + 1)^2$ to examine limit at $x = -2$.

Solution. From the graph of the function $f(x) = 1/(x + 2)$ in the Figure 1.10(a), one can conclude that $\lim_{x\to -2^+} f(x) = \infty$ and $\lim_{x\to -2^-} f(x) = -\infty$. Since the left hand and right hand limits are different we can conclude that $\lim_{x\to -2} f(x)$ does not exist.

From the graph of the function $f(x) = 1/(x + 2)^2$ in the Figure 1.10(b), one can conclude that $\lim_{x\to -2^+} f(x) = \infty$ and $\lim_{x\to -2^-} f(x) = \infty$. Since the left hand and right hand limits, both tend to ∞, we conclude that $\lim_{x\to -2} f(x) = \infty$.

Definition 1.5. **(Formal Definition of Infinite Limits)**

Let $f(x)$ be a function defined on an interval containing a, except possibly at a. We say that $f(x)$ approaches to ∞ as x approached to a, and denotes it as:



(a) $f(x) = 1/(x+2)$ (b) $f(x) = 1/(x+2)^2$

Figure 1.10: Infinite Limits at $x = -2$

$$\lim_{x \to a} f(x) = \infty$$

if, given any positive number B however large, there exists a $\delta > 0$, such that:

$$f(x) > B \quad whenever \quad 0 < |x - a| < \delta$$

Similarly, We say that $f(x)$ approaches to $-\infty$ as x approached to a, and denotes it as:

$$\lim_{x \to a} f(x) = -\infty$$

if, given any negative number $-B(B > 0)$ however small, there exists a $\delta > 0$, such that:

$$f(x) < -B \quad whenever \quad 0 < |x - a| < \delta$$

Example 1.13. Use the formal definition of limit to show that $\lim_{x \to 0} 1/x^2 = \infty$.

Solution. We need to show that given any $B > 0$ (however large), $\exists\, \delta > 0$, such that:
$f(x) > B$ whenever $0 < |x - a| < \delta$.
Here $f(x) = 1/x^2, a = 0$.
Now $\dfrac{1}{x^2} > B \iff x^2 < \dfrac{1}{B} \iff |x| < \dfrac{1}{\sqrt{B}}$
Therefore, taking $\delta = \dfrac{1}{\sqrt{B}}$, we have $\dfrac{1}{x^2} > B$ whenever $|x - 0| < \delta$
Hence, by definition 1.5, $\lim_{x \to 0} f(x) = \infty$.

1.7 Computing Limits

In this section, we will discuss the algebra of limits of functions. We will discuss the limits of the sum, product, and quotient of functions and use the results to obtain limits of various functions.

Theorem 1.3. ***(Properties of Limits)***

Let $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$*, where* L, M *are finite real numbers and* c *be any real number then the following holds:*

(a) $\lim_{x\to a} c = c.$

(b) $\lim_{x\to a} x = a.$

(c) $\lim_{x\to a}[c.f(x)] = c.\lim_{x\to a} f(x) = c.L$

(d) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$

(e) $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = L.M$

(f) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$*, provided* $M \neq 0.$

that is limit of the quotient of two functions equals to the quotient of the limits, provided the denominator limit is not equal to 0.

Note. If $L \neq 0$ and $M = 0$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ does not exist and will be equal to ∞ or $-\infty$.

If $L = M = 0$, then we obtain $(0/0)$, indeterminate form of limit, which we will discuss in Lesson-8.

(g) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$*, provided* $L > 0$ *if* n *is even.*

Note that $L > 0$ *is required only if* n *is even.*

These statements are also true for right hand $(x \to a^+)$ *and left hand* $(x \to a^-)$ *limits.*

Result 1.1. One can easily derive the following results by using the Theorem 1.3.

(a) $\lim_{x\to a}(f(x))^n = \underbrace{\lim_{x\to a} f(x).\lim_{x\to a} f(x)\ldots\lim_{x\to a} f(x)}_{n\,times} = (\lim_{x\to a} f(x))^n.$

(b) $\lim_{x\to a} x^n = \lim_{x\to a}\underbrace{(x.x....x)}_{n\,times} = \underbrace{\lim_{x\to a} x.\lim_{x\to a} x\ldots\lim_{x\to a} x}_{n\,times} = \underbrace{a.a.\ldots.a}_{n\,times} = a^n.$

(c) For any polynomial $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, we have:
$\lim_{x\to a} P(x) = \lim_{x\to a}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = a_0 \lim_{x\to a} 1 + a_1 \lim_{x\to a} x + a_2 \lim_{x\to a} x^2 + \cdots + a_n \lim_{x\to a} x^n = a_0 + a_1a + a_2a^2 + \cdots + a_na^n = P(a)$.

Example 1.14. Find $\lim_{x\to 2}(x^2 - 5x + 1)$.

Solution.

$$\begin{aligned}\lim_{x\to 2}(x^2 - 5x + 1) &= \lim_{x\to 2} x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 1 (\text{by using Theorem 1.3(b, c)})\\ &= 2^2 - 5 * 2 + 1 (\text{by using Theorem 1.3(a)})\\ &= -5\end{aligned}$$

(One can also use directly the Result 1.1(c))

In-text Exercise 1.2. Find $\lim_{x\to 3}(x^2 + 2x - 1)^5$.

Example 1.15. Find $\lim_{x\to 4} \frac{5x^2 + 6}{x - 5}$.

Solution.

$$\begin{aligned}\lim_{x\to 4} \frac{5x^2 + 6}{x - 5} &= \frac{\lim_{x\to 4}(5x^2 + 6)}{\lim_{x\to 4}(x - 5)} (\text{by using Theorem 1.3(f)})\\ &= \frac{5.4^2 + 6}{4 - 5} \text{ (here } M = \lim_{x\to 4}(x - 5) = 4 - 5 = -1 \neq 0)\\ &= -86\end{aligned}$$

Example 1.16. Find the following limits:

$$(a) \lim_{x\to 5^+} \frac{x - 3}{x - 5} \qquad (b) \lim_{x\to 5^-} \frac{x - 3}{x - 5} \qquad (c) \lim_{x\to 5} \frac{x - 3}{x - 5}$$

Solution. We note that in each of the parts (a), (b), and (c), the denominator has limit 0, and the numerator has limit 2. So the limit does not exist in all the parts.

The limit is $+\infty$ or $-\infty$ depending on the sign. The numerator $x - 3$ remains positive, and the denominator $x - 5$ takes positive or negative value depending on x approached to 5, from right or left.

Thus we can conclude that: $\lim_{x\to 5^+} \frac{x - 3}{x - 5} = +\infty$ and $\lim_{x\to 5^-} \frac{x - 3}{x - 5} = -\infty$.

Since the left-hand limit and right-hand limit are different, so the two-sided limit at $x = 5$ does not exist.

Example 1.17. Find $\lim_{x\to 2} \frac{x^2 - 4x + 4}{x - 2}$.

Solution. Note that the numerator $x^2 - 4x + 4$ and denominator $x - 2$, both has limit 0 as $x \to 2$. So it needs to investigate further.

Note $\lim_{x\to 2} \frac{x^2 - 4x + 4}{x - 2} = \lim_{x\to 2} \frac{(x - 2)^2}{x - 2} = \lim_{x\to 2}(x - 2) = 0$.

Example 1.18. Consider the piecewise function f defined as:

$$f(x) = \begin{cases} \dfrac{1}{x-1} & x < 1 \\ x^2 - 5 & 1 \le x \le 3 \\ \sqrt{x+13} & x > 3 \end{cases}$$

Find the following limits:

$$(a) \lim_{x \to 1} f(x) \qquad (b) \lim_{x \to 2} f(x) \qquad (c) \lim_{x \to 3} f(x)$$

Solution. (a) we have

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{1}{x-1} = -\infty$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x^2 - 5 = -4$

$\implies \lim\limits_{x \to 1} f(x)$ does not exist.

Solution. (b) f takes same value close to $x = 2$ on both sides. So

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2}(x^2 - 5) = -1$

Solution. (c)

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (x^2 - 5) = 4$

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} \sqrt{x+13} = \sqrt{16} = 4.$

$\implies \lim\limits_{x \to 3} f(x) = 4.$

Example 1.19. Explain why we can not apply Theorem 1.3(d) to evaluate $\lim\limits_{x \to 0}(\dfrac{1}{x} - \dfrac{1}{x^2})$.

Solution. Since $\lim\limits_{x \to 0^-} \dfrac{1}{x} = -\infty$ and $\lim\limits_{x \to 0^+} \dfrac{1}{x} = \infty$, therefore $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist. Similarly $\lim\limits_{x \to 0} \dfrac{1}{x^2} = \infty$, which is not finite. Therefore we can not apply Theorem 1.3(d) to evaluate $\lim\limits_{x \to 0}(\dfrac{1}{x} - \dfrac{1}{x^2})$.

However one can proceed as follows to evaluate the limit:

$\lim\limits_{x \to 0}(\dfrac{1}{x} - \dfrac{1}{x^2}) = \lim\limits_{x \to 0}(\dfrac{x-1}{x^2})$. Now $\lim\limits_{x \to 0}(x - 1) = -1$ and $\lim\limits_{x \to 0} x^2 = 0$. Therefor $\lim\limits_{x \to 0}(\dfrac{1}{x} - \dfrac{1}{x^2}) = -\infty$, by the Note of Theorem 1.3(f).

1.8 Limit at Infinity

Let us examine the limit of a function when x increases or decreases without bounds. We write $x \to \infty$ to denotes x increases without bound and $x \to -\infty$ to denotes x decreases without bound. If the value of a function $f(x)$ can be made as close to L as we want by increasing or decreasing x, without a bound, then we denote it as the following:

$$\lim_{x\to\infty} f(x) = L \quad \text{and} \lim_{x\to-\infty} f(x) = L$$

.

Definition 1.6. (Limit at Infinity)

Let f be a function defined for $x \geq a$ for some a.

We say that $f(x)$ approaches to L as x approaches to ∞, and denotes as

$$\lim_{x\to\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a $M > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M.$$

Similarly, we say that f approaches to L as x approaches to $-\infty$, and denotes as

$$\lim_{x\to-\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a $N < 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x < N$$

.

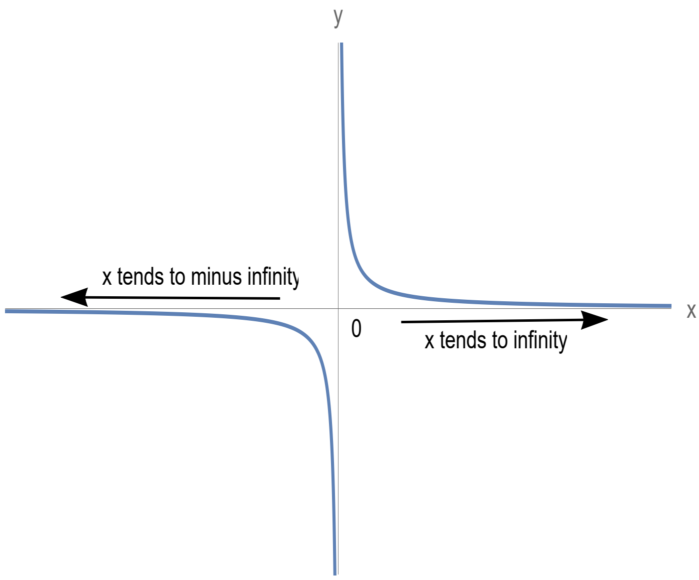


Figure 1.11: Graph of the function $f(x) = 1/x$.

Example 1.20. Show that:

$$(a) \lim_{x\to\infty} \frac{1}{x} = 0 \qquad (b) \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Solution. (a) Here $f(x) = 1/x, L = 0$

We need to show for any $\epsilon > 0,$ there exists $M > 0$, such that $|f(x)-L| < \epsilon \quad$ whenever $\quad x > M.$

$$\begin{aligned}
\text{Now} \qquad |f(x) - L| &< \epsilon \\
\text{whenever } |1/x - 0| &< \epsilon \\
\text{or } 1/x &< \epsilon \quad \text{we can take } x > 0 \\
\text{or } x &> 1/\epsilon = M(\text{say})
\end{aligned}$$

$$\implies |1/x - 0| < \epsilon \quad \text{whenever} \quad x > M$$
$$\implies \lim_{x\to\infty} \frac{1}{x} = 0.$$

(b) Here $f(x) = 1/x, L = 0$

We need to show for any $\epsilon > 0,$ there exists $N < 0$, such that $|f(x)-L| < \epsilon \quad$ whenever $\quad x < N.$

$$\begin{aligned}
\text{Now} \qquad |f(x) - L| &< \epsilon \\
\text{whenever } |1/x - 0| &< \epsilon \\
\text{or } \; -1/x &< \epsilon \quad \text{as } x < 0 \text{ for } x \to -\infty \\
\text{or } \; -x &> 1/\epsilon \\
\text{or } \; x &< -1/\epsilon = N(\text{say, } N \text{is negative})
\end{aligned}$$

$$\implies |1/x - 0| < \epsilon \quad \text{whenever} \quad x < N$$
$$\implies \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Theorem 1.4. ***(Properties of Limits at infinity)***

Let $\lim\limits_{x\to\pm\infty} f(x) = L$ *and* $\lim\limits_{x\to\pm\infty} g(x) = M$*, and c be any real number then the following results hold:*

(a) $\lim\limits_{x\to\pm\infty} c = c.$

(b) $\lim\limits_{x\to\pm\infty} x = a.$

(c) $\lim\limits_{x\to\pm\infty} [c.f(x)] = c. \lim\limits_{x\to\pm\infty} f(x) = c.L$

(d) $\lim\limits_{x\to\pm\infty} [f(x) \pm g(x)] = \lim\limits_{x\to\pm\infty} f(x) \pm \lim\limits_{x\to\pm\infty} g(x) = L \pm M$

(e) $\lim\limits_{x\to\pm\infty} [f(x).g(x)] = \lim\limits_{x\to\pm\infty} f(x). \lim\limits_{x\to\pm\infty} g(x) = L.M$

(f) $\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to\pm\infty} f(x)}{\lim_{x\to\pm\infty} g(x)} = \dfrac{L}{M}$*, provided* $M \neq 0.$

(g) $\lim_{x\to\pm\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\pm\infty} f(x)} = \sqrt[n]{L}$*, provided* $L > 0$ *if* n *is even.*

Note that $L > 0$ *is required only if* n *is even.*

Standard Limit: Following is one of the standard limit at infinity.

$$\lim_{n\to\infty}(1+\frac{1}{x})^x = \mathrm{e} = (2.7182...)$$

Example 1.21. Find $\lim_{x\to\infty}\frac{1}{x^n}$, n is a positive integer.

Solution. $\lim_{x\to\infty}\frac{1}{x^n} = \lim_{x\to\infty}(\frac{1}{x})^n = (\lim_{x\to\infty}\frac{1}{x})^n$ = 0 (by Theorem 1.4(e) and Example 1.20)

Example 1.22. Evaluate $\lim_{x\to\infty}\frac{2x^2+3x+4}{4x^2+3x+2}$.

Solution. Note that as $x \to \infty$ numerator and denominator both tends to ∞, and we get an indeterminate form $\frac{\infty}{\infty}$, which we will discuss in Lesson-8. In this case, therefore, alternatively, we can divide the numerator and denominator by the highest power of x in the denominator to evaluate the limit.

$\lim_{x\to\infty}\frac{2x^2+3x+4}{4x^2+3x+2} = \lim_{x\to\infty}\frac{2+3/x+4/x^2}{4+3/x+2/x^2} = \frac{2+0+0}{4+0+0} = 2/4 = 1/2.$ (by using Theorem 1.4)

Example 1.23. Evaluate $\lim_{x\to\infty}\frac{5x^2+2x+1}{4x^3+2x+1}$.

Solution. Dividing the numerator and denominator by the highest power of x in the denominator, $i.e.$ by x^3, we get

$\lim_{x\to\infty}\frac{5x^2+2x+1}{4x^3+2x+1} = \lim_{x\to\infty}\frac{5/x+2/x^2+1/x^3}{4+2/x^2+2/x^3} = 0/4 = 0$ (by using Theorem 1.4).

In-text Exercise 1.3. Find $\lim_{x\to\infty}\frac{\sqrt{3x^4+7x+5}}{4x^2}$.

1.9 Infinite limit at infinity

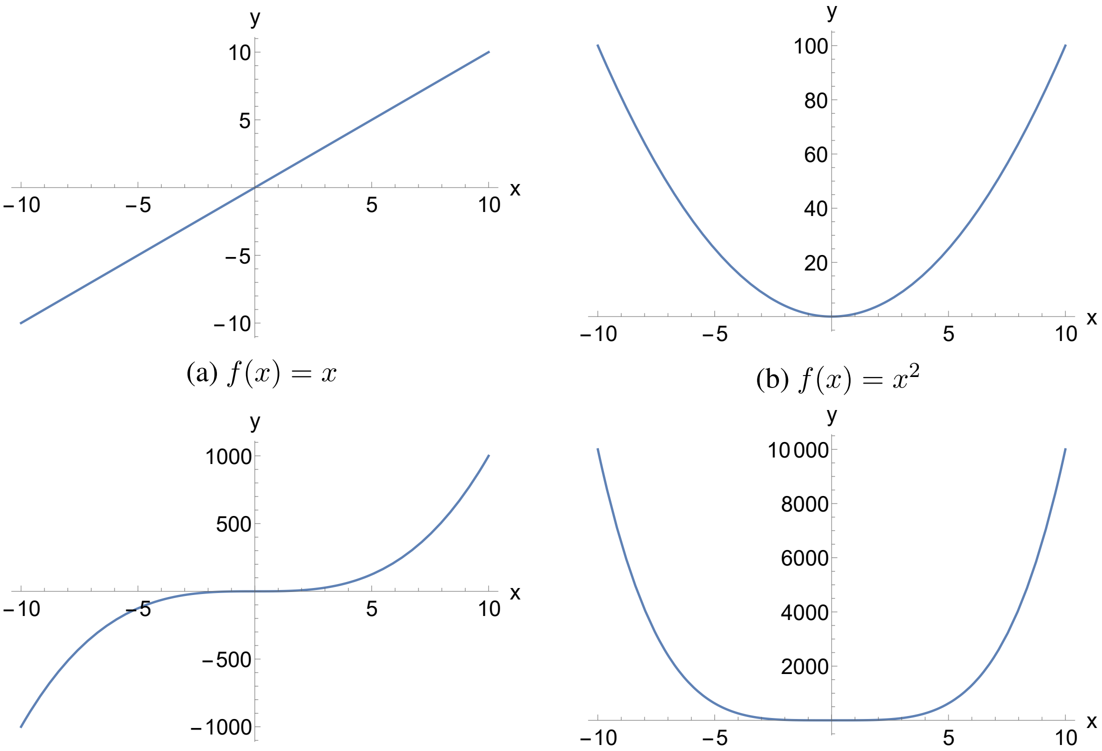
Definition 1.7. (**Infinite limit at infinity**)

If the function increases without bound when x approached to ∞ or $-\infty$, then we denote this limit as:

$$\lim_{x\to\infty} f(x) = \infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = \infty$$

Similarly, if the function decreases without bound when x approached to ∞ or $-\infty$, then we denote this limit as:

$$\lim_{x\to\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = -\infty$$



(a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = x^3$

(d) $f(x) = x^4$

Figure 1.12: Graphs of x^n

For illustration, from the graphs of the functions $f(x) = x^n (n = 1, 2, 3, 4)$ shown in Figure 1.12, one can obtain the following:

$$\lim_{x \to \infty} x = \infty, \lim_{x \to -\infty} x = -\infty$$

$$\lim_{x \to \infty} x^2 = \infty, \lim_{x \to -\infty} x^2 = \infty$$

$$\lim_{x \to \infty} x^3 = \infty, \lim_{x \to -\infty} x^3 = -\infty$$

$$\lim_{x \to \infty} x^4 = \infty, \lim_{x \to -\infty} x^4 = \infty$$

Result 1.2. One can obtain the following:

1.
$$\lim_{x \to \infty} x^n = \infty, n = 1, 2, 3, \dots$$

2.
$$\lim_{x \to -\infty} x^n = \begin{cases} -\infty & \text{if}\, n = 1, 3, 5, \dots \\ \infty & \text{if}\, n = 2, 4, 6, \dots \end{cases}$$

3. The limit of a polynomial $P(x)$ as $x \to \pm\infty$ is same as the limit of its highest degree term as $x \to \pm\infty$. That is, if $P(x) = (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)$, then

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \lim_{x\to\infty} a_n x^n.$$

$$\lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = \lim_{x\to-\infty} a_n x^n.$$

Example 1.24. Find $\lim_{x\to\infty} (5x^4 + 3x^2 - 4x + 1)$.

Solution. $\lim_{x\to\infty} (5x^4 + 3x^2 - 4x + 1) = \lim_{x\to\infty} 5x^4$ (using Result 1.2(3)).
$= \infty$.

Limit of Rational function at $x = \pm\infty$

The limit of a rational function at $x = \pm\infty$ can be obtained by dividing the numerator and denominator by the highest power of x that appears in the denominator.

Example 1.25. Find $\lim_{x\to-\infty} \dfrac{5x^3 - 4x + 2}{3x^2 + 2x + 1}$

Solution.

$$\begin{aligned}\lim_{x\to-\infty} \frac{5x^3 - 4x + 2}{3x^2 + 2x + 1} &= \lim_{x\to-\infty} \frac{5x - 4/x + 2/x^2}{3 + 2/x + 1/x^2} \quad \text{(dividing by } \quad x^2) \\ &= \frac{\lim_{x\to-\infty} 5x - 4/x + 2/x^2}{\lim_{x\to-\infty} 3 + 2/x + 1/x^2} \\ &= \frac{\lim_{x\to-\infty} 5x}{\lim_{x\to-\infty} 3} \\ &= -\infty/3 = -\infty\end{aligned}$$

Result 1.3. (Limit of Trigonometric, Exponential, and Logarithmic functions at infinity) From the graphs of some standard trigonometric, exponential and logarithmic functions shown in Figure 1.13, one can conclude the following:

1. $\lim_{x\to\infty} sin(x)$ does not exist.
2. $\lim_{x\to\infty} cos(x)$ does not exist.
3. $\lim_{x\to\infty} tan(x)$ does not exist.
4. $\lim_{x\to\infty} e^x = \infty$.
5. $\lim_{x\to-\infty} e^x = 0$.
6. $\lim_{x\to\infty} e^{-x} = 0$

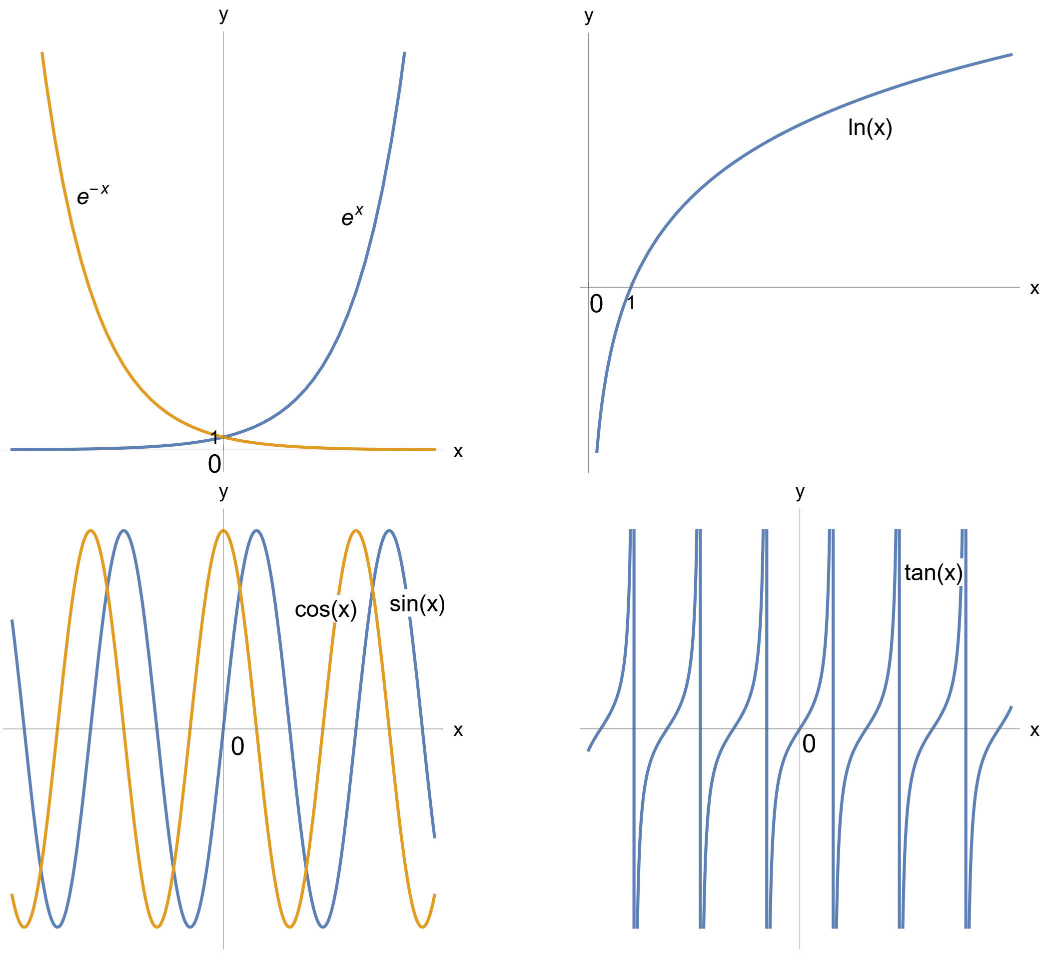


Figure 1.13: Graphs of some standard functions.

7. $\lim_{x \to -\infty} e^{-x} = \infty.$

8. $\lim_{x \to \infty} e^{1/x} = 1.$

9. $\lim_{x \to -\infty} e^{1/x} = 1.$

10. $\lim_{x \to 0^+} e^{-1/x} = 0.$

11. $\lim_{x \to 0^-} e^{1/x} = 0.$

12. $\lim_{x \to \infty} ln(x) = \infty.$

13. $\lim_{x \to 0^+} ln(x) = -\infty.$

In-text Exercise 1.4. Find the limit $\lim_{x \to -\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

In-text Exercise 1.5. Find the limit $\lim_{x\to\infty}(1+\frac{1}{x})^{2x}$.

In-text Exercise 1.6. Find the limit $\lim_{x\to 0}\frac{e^{1/x}+1}{e^{1/x}-1}$.

1.10 Summary

1. If the function $f(x)$ takes values close to L, by taking values of x sufficiently close to a, then we say that $f(x) \to L \quad as \quad x \to a$.

2. If function $f(x)$ takes values close to L, by taking values of x sufficiently close to a from the right side, that is, for $x > a$, then we say $f(x) \to L \quad as \quad x \to a^+$.

3. If function $f(x)$ takes values close to L, by taking values of x sufficiently close to a from the left side, that is, for $x < a$, then we say $f(x) \to L \quad as \quad x \to a^-$.

4. If f is a function defined on an open interval containing a, but not necessarily at a. We say $\lim_{x\to a} f(x) = L$, if given any $\epsilon > 0$, there exists a $\delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

.

5. We say that $f(x)$ approaches to infinity as x approached to a, and denotes as:

$$\lim_{x\to a} f(x) = \infty$$

if, given any positive number B however large, there exists a $\delta > 0$, such that:

$$f(x) > B \quad \text{whenever} \quad 0 < |x - a| < \delta$$

6. We say that the function f approaches to L as x approaches to ∞, and denoted as

$$\lim_{x\to\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a $M > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M.$$

7. We say that the function f approaches to L as x approaches to $-\infty$, and denoted as

$$\lim_{x\to-\infty} f(x) = L$$

if for every $\epsilon > 0$, there exists a $N < 0$, such that:

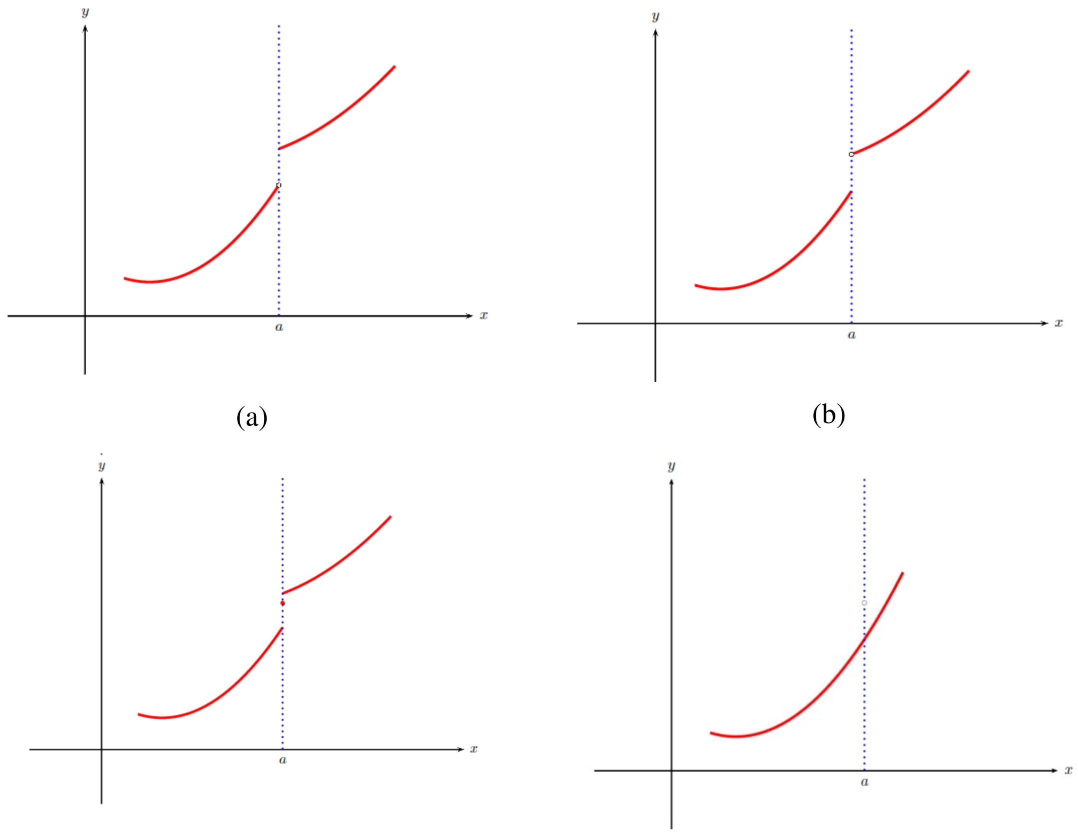
$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x < N$$

.

1.11 Self Assessment Exercises

1. Use numerical values and plot of the function to find $\lim_{x\to 0} sin(x) = 0$.
2. Use numerical values and plot of the function to find $\lim_{x\to 0} |x|$.
3. Use numerical values and plot of the function to find $\lim_{x\to 1}(x^2 + 1)$.
4. Use numerical values and plot of the function to find $\lim_{x\to 0} cos(\pi/x)$.
5. Show that the limit $\lim_{x\to 1} \frac{|x-1|}{x-1}$ does not exist.
6. Show that the limit $\lim_{x\to 0} \frac{x-|x|}{x}$ does not exist.
7. Show that the limit $\lim_{x\to 0} \frac{xe^{1/x}}{1+e^{1/x}} = 0$
8. Find the limit $\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$.
9. Find $\lim_{x\to\infty} (1 + \frac{1}{x})^{-x}$.
10. Find $\lim_{x\to\infty} ln(\frac{1}{x^2})$.
11. Use numerical values and plot of the function to find $\lim_{x\to 0} xSin(1/x)$.
12. Use numerical values and graph of the function to show that $\lim_{x\to 0} Sin(x)/x = 1$.
13. Use $\epsilon - \delta$ definition to show that $\lim_{x\to 0} xCos(1/x) = 0$.
14. Use $\epsilon - \delta$ definition to show that $\lim_{x\to 2}(3x - 5) = 1$.
15. For $f(x) = 2x + 3, L = 3$, find the value of δ such that $|f(x) - L| < \epsilon$, whenever $|x - 0| < \delta$, for each of the following ϵ:

    (i) $\epsilon = 0.5$ *(ii)* $\epsilon = 0.2$, *(iii)* $\epsilon = 0.1$
16. Find $\lim_{x\to a} \frac{|x-a|}{x-a}$.
17. For the graphs of the functions shown in Figure 1.14, discuss whether the left hand limit, right hand limit, and the limit of the functions at $x = a$, exist or not.



(a) (b) (c) (d)

Figure 1.14: Graphs

1.12 Solution of In-text Exercises

In-text Exercise 1.1

Solution. Here we have $f(x) = 4x - 3, L = 5, a = 2$. We need to show that given any $\epsilon > 0, \exists \quad \delta > 0$, such that:

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < \quad |x - a| < \delta$$

that is $|f(x) - 5| < \epsilon \quad$ whenever $\quad 0 < |x - 2| < \delta$.

$$\begin{aligned} \text{Now} \quad & |f(x) - L| < \epsilon \\ \text{whenever} \quad & |4x - 3 - 5| < \epsilon \\ \text{or} \quad & |4x - 8| < \epsilon \\ \text{or} \quad & 4|x - 2| < \epsilon \\ \text{or} \quad & |x - 2| < \epsilon/4 \end{aligned}$$

Therefore for $\epsilon > 0$, there exists $\delta = \epsilon/4$, such that $|f(x) - 5| < \epsilon$ whenever $|x - 2| < \delta(= \epsilon/4)$.

Hence by definition 1.3, $\lim_{x \to 2}(4x - 3) = 5$.

In-text Exercise 1.2

Solution.

$$\begin{aligned}\lim_{x\to 3}(x^2+2x-1)^5 &= (\lim_{x\to 3}(x^2+2x-1))^5 \text{(using Result 1.3(a))}\\ &= (3^2+2*3-1)^5 \text{(using Result 1.3(b, c))}\\ &= 14^5 = 537824\end{aligned}$$

In-text Exercise 1.3

Solution. $\lim_{x\to\infty} \frac{\sqrt{3x^4+7x+5}}{4x^2} = \lim_{x\to\infty} \frac{1}{4}\sqrt{3+\frac{7}{x^3}+\frac{5}{x^4}} = \frac{\sqrt{3}}{4}$.

In-text Exercise 1.4

Solution. $\lim_{x\to-\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} = \lim_{x\to-\infty} \frac{e^{2x}+1}{e^{2x}-1} = \frac{0+1}{0-1} = -1$.

In-text Exercise 1.5

Solution. $\lim_{x\to\infty} (1+\frac{1}{x})^{2x} = \lim_{x\to\infty} ((1+\frac{1}{x})^x)^2 = e^2$.

In-text Exercise 1.6

Solution. $\lim_{x\to 0-} \frac{e^{1/x}+1}{e^{1/x}-1} = \frac{0+1}{0-1} = -1$.

$\lim_{x\to 0+} \frac{e^{1/x}+1}{e^{1/x}-1} = \lim_{x\to 0+} \frac{1+e^{-1/x}}{1-e^{-1/x}} = \frac{1+0}{1-0} = 1$.

Thus $\lim_{x\to 0} \frac{e^{1/x}+1}{e^{1/x}-1}$ does not exists.

1.13 Suggested Readings

1. Anton, Howard, Bivens, Irl, Davis, Stephen (2013). Calculus (10th ed.). Wiley India Pvt. Ltd. New Delhi. International Student Version. Indian Reprint 2016.

2. Prasad, Gorakh (2016). Differential Calculus (19th ed.). Pothishala Pvt. Ltd. Allahabad

3. Thomas Jr., George B., Weir, Maurice D., Hass, Joel (2014). Thomas' Calculus (13thed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 2

Continuity

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Structure

2.1 Learning Objectives

1. To understand the concept of continuity.
2. To learn the properties of a continuous function.
3. To learn the types of discontinuities.
4. To learn about the behavior of various discontinuous functions at the point of discontinuity.

2.2 Introduction

In Lesson 1, we studied the limit of a function. The criterion for the existence of $\lim_{x \to a} f(x)$ does not require the function to be defined at the point $x = a$, even if it is defined at $x = a$, the limit may not be equal to the value of the function at $x = a$. The existence of limit only

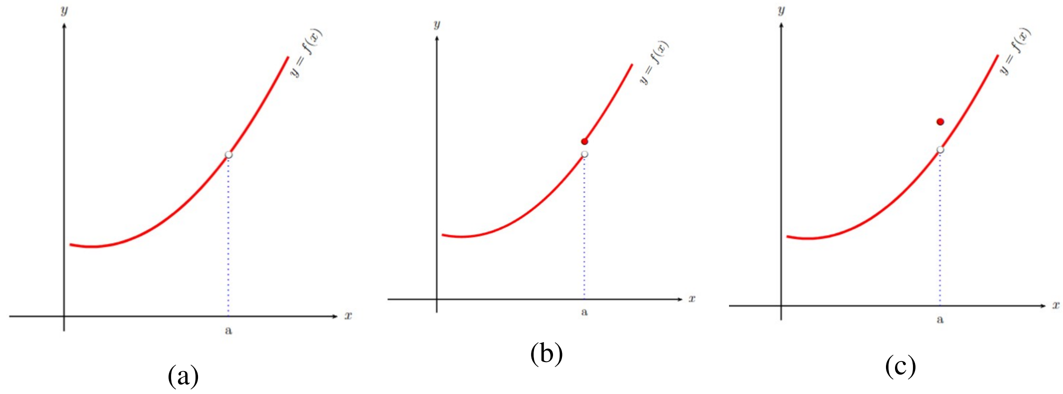

Figure 2.1: Impossible trajectory of a projectile

ensures that the function does not change abruptly or vanish suddenly around the point $x = a$. In real life, the projectile of any object thrown in air can not vanish at some point and suddenly reappear. For example, the graphs of function $f(x)$ shown in Figure 2.1 can not be the trajectory of a projectile. To capture the motion of a projectile mathematically, we need the concept of continuity. In this lesson, we will study continuous functions and their properties.

2.3 Continuous Function

Definition 2.1. (**Continuous Function**)
A function f defined at a point $x = a$ and around it, is called continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

From above definition, we note that $f(x)$ is continuous at $x = a$, if and only if following conditions are satisfied:

1. f is defined at $x = a$.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

Definition 2.2. (**Discontinuous Function**)
A function f is said to be discontinuous at $x = a$ if it is not continuous at $x = a$. Thus f is discontinuous at $x = a$, if one of the above condition does not hold.

Note. For illustration, the functions shown in Figure 2.1 are not continuous at $x = a$, because

1. In part(a) the function is not defined at point $x = a$.

2. In part(b) $\lim_{x \to a} f(x)$ does not exists.

3. In part(c) $\lim_{x \to a} f(x)$ exists, but not equal to the function value.

Example 2.1. Discuss the continuity of the following functions at $x = 3$:

(a) $f(x) = \dfrac{x^2 - 9}{x - 3}$.

(b) $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if} \quad x \neq 3 \\ 4 & \text{if} \quad x = 3 \end{cases}$

(c) $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if} \quad x \neq 3 \\ 6 & \text{if} \quad x = 3 \end{cases}$

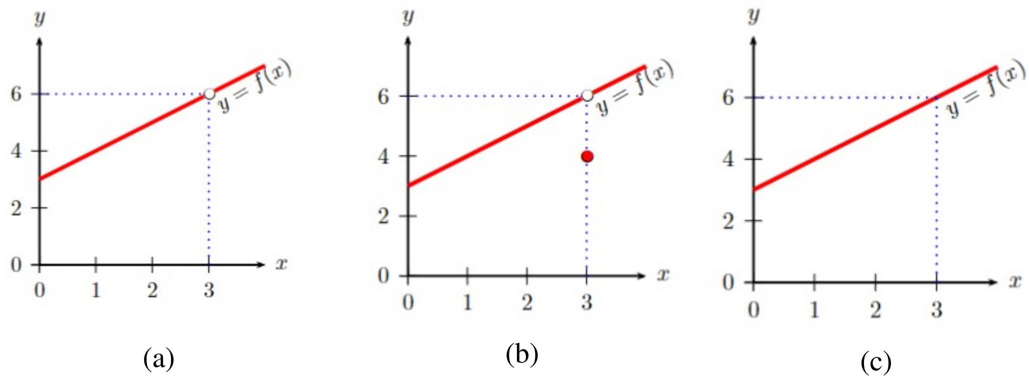

Figure 2.2: Graphs of $f(x)$.

Solution. The graphs of the functions in parts (a), (b), (c) are shown in Figure 2.2.
(a) The given function is not defined at $x = 3$, so it is discontinuous at $x = 3$.
(b) Lets first find out the limit of the function at $x = 3$.

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3}(x + 3) = 6.$$

As, $\lim_{x \to 3} f(x) = 6 \neq f(3) = 4$, the given function is discontinuous at $x = 3$.
(c) $\lim_{x \to 3} f(x) = 6 = f(3)$, the given function is continuous at $x = 3$.

In-text Exercise 2.1. Check the continuity of the function

$$f(x) = \begin{cases} \frac{xe^{1/x}}{1+e^{1/x}} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

at $x = 0$.

Definition 2.3. **(Continuity on an open interval)**

A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at each point $x \in (a, b)$.

Note. Note that the above definition is applied even to the infinite open intervals (∞, b), (a, ∞), $(-\infty, +\infty)$. If a function is continuous on $(-\infty, +\infty)$, then we say that f is continuous everywhere.

It is not possible to extend the above definition of continuity to the close interval $[a.b]$, as we can not talk about left hand limit at a and right hand limit at b.

Definition 2.4. **(Continuity on a close interval)**

A function $f(x)$ is said to be continuous on an close interval $[a, b]$, if following conditions holds:

1. f is continuous on the open interval (a, b).

2. f is continuous from the right at a, that is, $\lim_{x\to a^+} f(x) = f(a)$.

3. f is continuous from the left at b, that is, $\lim_{x\to b^-} f(x) = f(b)$.

Example 2.2. Discuss the continuity of the function $f(x) = \sqrt{4 - x^2}$.

Solution. The given function $f(x) = \sqrt{4 - x^2}$ defined (takes real values) on the close interval $[a, b]$. So we need to check the continuity of the given function on the open interval $(-2, 2)$ and at the endpoints $x = -2, 2$.

If c is any point in the open interval $(-2, 2)$, then

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c} \sqrt{4 - x^2} \\
&= \sqrt{\lim_{x\to c}(4 - x^2)} \quad \text{using Theorem 1.3(g)} \\
&= \sqrt{4 - c^2} = f(c)
\end{aligned}$$

$$\implies f(x) \text{ is continuous on any } c \in (-2, 2).$$

Now $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \sqrt{4 - x^2} = \sqrt{\lim_{x\to -2^+} (4 - x^2)} = 0 = f(-2)$

Also $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sqrt{4 - x^2} = \sqrt{\lim_{x\to 2^-} (4 - x^2)} = 0 = f(2)$

$\implies$ f is continuous at $x = -2$ and 2.

Thus, f is continuous on the close interval $[-2, 2]$.

Theorem 2.1. ***(Algebra of Continuous Functions)***

Let f be g be two functions continuous at $x = a$. Then the following hold:

1. *$f + g$ is continuous at a.*

2. *$f - g$ is continuous at a.*

3. *$f * g$ is continuous at a.*

4. *f/g is continuous at a if $g(a) \neq 0$, and discontinuous at a if $g(a) = 0$.*

Result 2.1. (a) A constant function is continuous everywhere (continuous on $\mathbb{R}$).

(b) A polynomial function is continuous everywhere.

(c) A rational function $f(x) = \dfrac{P(x)}{Q(x)}$ (where $P(x), Q(x)$ are polynomials) is continuous everywhere except at the points where $Q(x)$ equlas to zero.

Example 2.3. Discuss the continuity of the following function on the interval $[\,0, 1]$.

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{1}{2} - x & \text{if} \quad 0 < x < \frac{1}{2} \\ 0 & \text{if} \quad x = \frac{1}{2} \\ x - \frac{1}{2} & \text{if} \quad \frac{1}{2} < x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

Solution. Note that the function changes its behavior at the points $x = 0, 1/2, 1$. Therefore, these are the possible points of discontinuity. The function being a polynomial is continuous on the interval $(0, 1/2)$ and $(1/2, 1)$.

Continuity at $x = 0$.

$x = 0$ being a left end point of the interval [0, 1], the continuity at point $x = 0$ requires only continuity from right, that is, we need to check whether $\lim\limits_{x\to 0^+} f(x) = f(0) = 0$.

we have, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (\dfrac{1}{2} - x) = 1/2 - 0 = 1/2 \neq f(0)(= 0)$.

Thus, f is not continuous at $x = 0$.

Continuity at $x = 1/2$.

To check the continuity at point $x = 1/2$ we need to check whether $\lim\limits_{x\to 1/2} f(x) = f(1/2) = 0$.

we have $\lim\limits_{x\to (1/2)^-} f(x) = \lim\limits_{x\to (1/2)^-} (1/2 - x) = 1/2 - 1/2 = 0$,

and $\lim\limits_{x\to (1/2)^+} f(x) = \lim\limits_{x\to (1/2)^+} x - 1/2 = 1/2 - 1/2 = 0$.

this implies $\lim\limits_{x\to 1/2} f(x) = 0 = f(1/2)$, so f is continuous at $x = 1/2$.

Continuity at $x = 1$.

$x = 1$, being a right end point of the interval [0, 1], the continuity at point $x = 0$ requires only continuity from left, that is we need to check whether $\lim\limits_{x\to 1-} f(x) = f(1) = 1$.

we have $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x - \dfrac{1}{2}) = 1 - 1/2 = 1/2 \neq f(1)(= 1)$.

Thus, f is not continuous at $x = 1$.

Example 2.4. Find the points where the following function is discontinuous:

$$f(x) = \frac{x^2 - 4}{x^2 - 7x + 10}$$

.

Solution. The given function is a rational function. So the points of discontinuities are given by the roots of the denominator (by Result 2.1(c)), which are given by the roots of $x^2 - 7x + 10 = 0$.

Therefore, $x = 2, 5$ are the point of discontinuity.

Example 2.5. Show that the function $f(x) = |x|$ is continuous everywhere.

Solution. Given

$$f(x) = |x| = \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

$f(x)$ being polynomial in the domain $x > 0$ and $x < 0$, is continuous for $x > 0$ and $x < 0$. The only possible point of discontinuity is $x = 0$.

Now $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} |x| = \lim_{x\to 0-} (-x) = 0$

$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} |x| = \lim_{x\to 0+} (x) = 0$

$\implies \lim_{x\to 0} f(x) = 0 = f(0)$, Hence $f(x) = |x|$ is continuous at $x = 0$.

This implies $|x|$ is continuous everywhere on $\mathbb{R}$.

Example 2.6. Find the values of a and b, for which the following function is continuous.

$$f(x) = \begin{cases} ax^2 + b & \text{if} \quad x \leq 0 \\ 1 - \frac{3}{x^2+1} & \text{if} \quad 0 < x \leq 1 \end{cases}$$

Solution. We have $f(0) = b$.

Since f is continuous at $x = 0$, this implies that at $x = 0$, left-hand limit and right-hand limit both exist and these are equal to $f(0)$.

Now

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} (ax^2 + b) \\ &= b \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} (1 - \frac{3}{x^2 + 1}) \\ &= 1 - 3 = -2. \end{aligned}$$

Therefore $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0), \implies b = -2.$

Since there is no condition on a, it can take any value.

So given function is continuous for any real value of a and $b = -2$.

In-text Exercise 2.2. Find the value of a for which the following function is continuous.

$$f(x) = \begin{cases} 2x-1 & \text{if} \quad x<2 \\ a & \text{if} \quad x=2 \\ x+1 & \text{if} \quad x>2 \end{cases}$$

Example 2.7. Check the continuity of the function $f(x) = |x| + |x-1|$ at $x = 0, 1$.

Solution. Given

$$f(x) = |x| + |x-1| = \begin{cases} -x + (-(x-1)) = 1-2x & \text{if} \quad x<0 \\ x + (-(x-1)) = 1 & \text{if} \quad 0 \le x < 1 \\ x + (x-1) = 2x-1 & \text{if} \quad x \ge 1 \end{cases}$$

Now, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (1-2x) = 1.$
and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 1 = 1.$
So, $\lim\limits_{x\to 0} f(x) = 1 = f(0)$
$\implies f$ is continuous at $x = 0$.
Also, $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 0^-} 1 = 1.$
and $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 0^+} (2x-1) = 2-1 = 1.$
So $\lim\limits_{x\to 1} f(x) = 1 = f(1)$
$\implies f$ is continuous at $x = 1$.

Theorem 2.2. *If a function f is continuous at L and g is a function such that $\lim\limits_{x\to a} g(x) = L$, then we have:*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(L)$$

Note: *The result still holds if we replace $\lim\limits_{x\to a}$ by $\lim\limits_{x\to a^-}$ or $\lim\limits_{x\to a^+}$, or $\lim\limits_{x\to -\infty}$, or $\lim\limits_{x\to \infty}$.*

Remark. In other words the above theorem tells that the limit symbol can be moved inside a function if the limit of the inside expression (of the function) exists, and the function is continuous at the limit of the inside expression.

Example 2.8. Show that $\lim\limits_{x\to 3} |6-x^2|$ exists.

Solution. We know that the function $f(x) = |x|$ is continuous everywhere. In particular f is continuous at $\lim\limits_{x\to 3}(6-x^2) = -3$.

So by the Theorem 2.2, $\lim\limits_{x\to 3} |6-x^2| = |\lim\limits_{x\to 3}(6-x^2)| = |-3| = 3$.

Theorem 2.3. ***(Continuity of the Composition of Functions)***

Let $X, Y \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are functions such that $f(X) \subseteq Y$. If f is continuous at $a \in X$ and g is continuous at $b = f(a) \in Y$, then the composite function $gof : X \to \mathbb{R}$ defined as:

$$gof(x) = g(f(x))$$

is also continuous at a.

Result 2.2. In general, if functions f and g are continuous everywhere, then the composition gof is continuous everywhere.

Example 2.9. Examine the continuity of the function $h(x) = Sin(x^2)$.

Solution. Let $f(x) = x^2$ and $g(x) = Sin(x)$, then the given function is a composition of these two functions, that is $gof(x) = g(f(x)) = g(x^2) = Sin(x^2)$.

Since the function $f(x) = Sin(x)$ and $g(x) = x^2$ are continuous everywhere, the composition function $gof(x)$ is continuous everywhere, by Result 2.2.

Hence the given function $h(x) = Sin(x^2)$ is continuous everywhere.

In-text Exercise 2.3. Discuss the continuity of the function $|x^4 + 3x^2 - 1|$ on $\mathbb{R}$.

2.4 Types of Discontinuity

Removable Discontinuity

A function f has removable discontinuity at $x = a$, if $\lim_{x \to a} f(x)$ exists but it is not equal to $f(a)$.

Such a discontinuity is called removable because it can be removed by redefining the function at $x = a$, that is by assigning $f(a) = \lim_{x \to a} f(x)$.

Discontinuity of First Kind

A function f has discontinuity of first kind at $x = a$, if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exists but these are unequal. That is:

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

This is also known as a Jump Discontinuity.

Discontinuity of First Kind from Right: A function f has discontinuity of first kind from right at $x = a$, if

$$\lim_{x \to a^+} f(x) \neq f(a) \text{ but } \lim_{x \to a^-} f(x) = f(a)$$

Discontinuity of First Kind from Left: A function f has discontinuity of first kind from left at $x = a$, if

$$\lim_{x \to a^-} f(x) \neq f(a) \text{ but } \lim_{x \to a^+} f(x) = f(a)$$

Discontinuity of Second Kind

A function f has discontinuity of second kind at $x = a$, if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both do not exist.

Discontinuity of Second Kind from Right: A function f has discontinuity of second kind from right if $\lim_{x \to a^+} f(x)$ does not exist.

Discontinuity of Second Kind from Left: A function f has discontinuity of second kind from left if $\lim_{x \to a^-} f(x)$ does not exist.

Example 2.10. Discuss the type of discontinuity of the following function at $x = 0$:

$$f(x) = \begin{cases} sin(1/x) & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

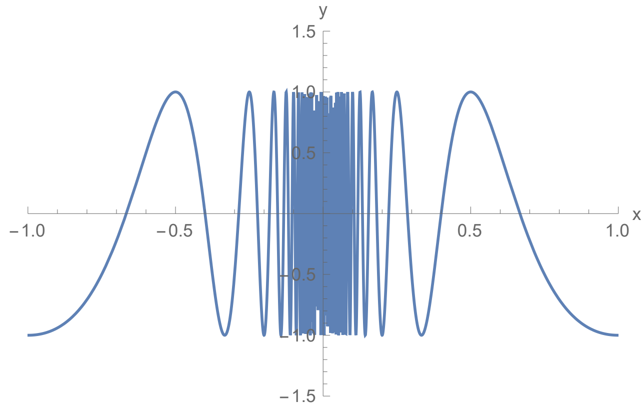

Figure 2.3: Graph of $sin(1/x)$

Solution. From the graph of the given function in Figure 2.3, we can note that the function oscillated rapidly between -1 and 1 as we move closer to 0. So the left-hand limit and the right-hand limit do not exist. Hence the function has a discontinuity of the second kind at $x = 0$.

Example 2.11. Discuss the type of discontinuity of the following function at $x = 0$:

$$f(x) = \begin{cases} |x|/x & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Solution. We have $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} |x|/x = \lim\limits_{x \to 0^-} -x/x = -1.$

and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} |x|/x = \lim\limits_{x \to 0^+} x/x = 1.$

Since left-hand limit and right-hand limit both exist but they are unequal, the function has discontinuity of first kind (or jump discontinuity) at $x = 0$.

Example 2.12. Discuss the type of discontinuity of the following function at $x = 0$:

$$f(x) = \begin{cases} sin(x)/x & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Solution. From the graph of the given function in Figure 2.4, we can note that the function approaches to 1 as x tends to 0, that is $\lim\limits_{x \to 0} sin(x)/x = 1$.

Since $f(0) = 0 \neq \lim\limits_{x \to 0} \dfrac{Sin(x)}{x}$, this implies that f has a removable discontinuity at $x = 0$.

If we define another function $F(x)$ as

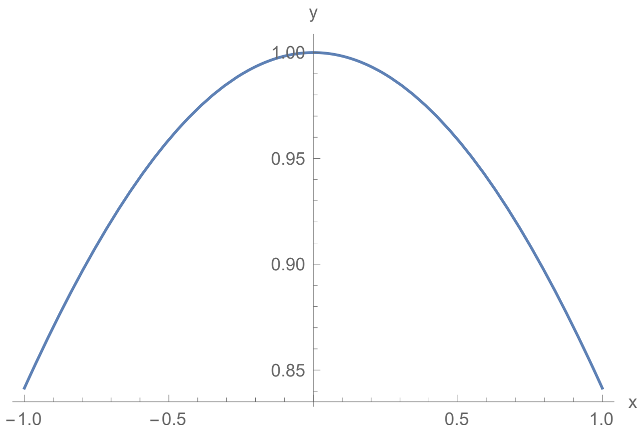

Figure 2.4: Graph of $sin(x)/x$

$$F(x) = \begin{cases} sin(x)/x & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

then $F(x)$ is continuous at $x = 0$.

Example 2.13. Consider the function defined as

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ x + 1/2 & \text{if} \quad 0 < x < 1/2 \\ 1/2 & \text{if} \quad x = 1/2 \\ x - 1/2 & \text{if} \quad 1/2 < x < 1 \\ 0 & \text{if} \quad x = 1. \end{cases}$$

Find the type of discontinuity if any.

Solution. The possible points of discontinuities are $x = 0, 1/2, 1$. In the domains $(0, 1/2)$ and $(1/2, 1)$ the given function is continuous being a polynomial.

At x = 0

We have $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x + 1/2) = 1/2$.

Since $\lim_{x \to 0^+} f(x) \neq f(0) = 1$, this implies that $x = 0$ is a point of discontinuity of the first kind from the right.

At x = 1/2

We have $\lim_{x \to (1/2)^-} f(x) = \lim_{x \to (1/2)^-} (x + 1/2) = 1$.

and $\lim_{x \to (1/2)^+} f(x) = \lim_{x \to (1/2)^+} (x - 1/2) = 0$.

Since, $\lim_{x \to (1/2)^-} f(x) \neq \lim_{x \to (1/2)^+} f(x)$, this implies $x = 1/2$ is a point of discontinuity of first kind.

At x = 1

We have $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 1/2) = 1/2$.

Since, $\lim_{x \to 1^-} f(x) \neq f(1)$ and $f(1) = 0$, this implies $x = 1$ is a point of discontinuity of first kind from the left.

2.5 Continuity of Some Standard Functions

Trigonometric Functions

If a is any number in the natural domain of the following standard trigonometric functions then we have

1. $\lim_{x \to a} sin(x) = sin(a)$, for all $a \in \mathbb{R}$.
2. $\lim_{x \to a} cos(x) = cos(a)$, for all $a \in \mathbb{R}$.
3. $\lim_{x \to a} tan(x) = tan(a)$, for all $a \in \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$.
4. $\lim_{x \to a} cot(x) = cot(a)$, for all $a \in \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$.
5. $\lim_{x \to a} csc(x) = csc(a)$, for all $a \in \mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$.
6. $\lim_{x \to a} sec(x) = sec(a)$, for all $a \in \mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$.

   The above results imply that the standard trigonometric functions are continuous in their natural domain.

   Exponential Function

7. $\lim_{x \to a} e^x = e^a$, for all $a \in \mathbb{R}$.

   That is, the exponential function is continuous everywhere.

   Logarithmic Function

8. $\lim_{x \to a} ln(x) = ln(a)$, for all $a > 0$.

   That is, the natural logarithmic function is continuous for $x > 0$.

Example 2.14. Find $\lim_{x \to 2} cos(\frac{x^2+5}{x+1})$.

Solution. Since $cos(x)$ is continuous everywhere, we have

$$\lim_{x \to 2} cos(\frac{x^2+5}{x+1}) = cos(\lim_{x \to 2} \frac{x^2+5}{x+1})$$
$$= cos(\frac{4+5}{2+1}) = cos(3).$$

In-text Exercise 2.4. Find $\lim_{x \to \pi} tan(2x)$.

2.6 Intermediate Value Theorems

Theorem 2.4. *(Intermediate Value Theorem)*

Let f be a function continuous on a closed interval $[a, b]$, and k be any number such that $f(a), \leq k \leq f(b)$, then there exists $x, a \leq x \leq b$, such that:

$$f(x) = k$$

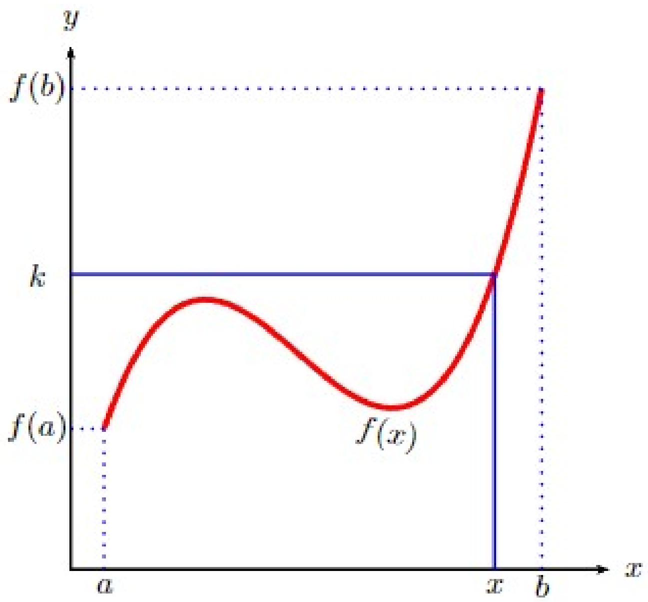

Figure 2.5: Graphical representation of Intermediate Value Theorem

A graphical representation of Intermediate Value theorem is shown in the Figure 2.5.

Theorem 2.5. *Let f be a function continuous on a closed interval $[a, b]$, and $f(a)$ and $f(b)$ are non-zero and they are of opposite signs, that is $f(a).f(b) < 0$, then there exists $x \in (a, b)$, such that $f(x) = 0$.*

Note that above theorem is a special case of Theorem 2.4, becasue as $f(a).f(b) < 0$, that is they are of opposite sign we have either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$. Here $k = 0$.

Example 2.15. Show that the function $f(x) = x^3 - x - 1$ has a root in the interval $(1, 2)$.

Solution. Given $f(x) = x^3 - x - 1$, we have $f(a) = f(1) = -1$ and $f(b) = f(2) = 5$.

This implies $f(1)f(2) < 0$, by theorem 2.5, $\exists x \in (1, 2)$, such that $f(x) = 0$. That is the equation has a root in the interval $(1, 2)$.

2.7 Summary

1. A function f defined at a point $x = a$ and around it, is called continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

2. A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at each point $x \in (a, b)$.

3. A function $f(x)$ is said to be continuous on an close interval $[a, b]$, if following conditions holds:

   (a) f is continuous on the open interval (a, b).

   (b) f is continuous from the right at a, that is, $\lim_{x \to a^+} f(x) = f(a)$.

   (c) f is continuous from the left at b, that is, $\lim_{x \to b^-} f(x) = f(b)$.

4. If $X, Y \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are functions such that $f(X) \subseteq Y$. If f is continuous at $a \in X$ and g is continuous at $b = f(a) \in Y$, then the function composition $gof : X \to \mathbb{R}$ defined as:

$$gof(x) = g(f(x))$$

   is also continuous at a.

5. A function f has removable discontinuity at $x = a$, if $\lim_{x \to a} f(x)$ exists but it is not equal to $f(a)$.

6. A function f has discontinuity of first kind from right at $x = a$, if

$$\lim_{x \to a^+} f(x) \neq f(a) \text{ but } \lim_{x \to a^-} f(x) = f(a)$$

7. A function f has discontinuity of first kind from left at $x = a$, if

$$\lim_{x \to a^-} f(x) \neq f(a) \text{ but } \lim_{x \to a^+} f(x) = f(a)$$

8. A function f has discontinuity of second kind at $x = a$, if $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both do not exist.

9. A function f has discontinuity of second kind from right if $\lim_{x \to a^+} f(x)$ does not exist.

10. A function f has discontinuity of second kind from left if $\lim_{x \to a^-} f(x)$ does not exist.

11. Let f be a function continuous on a closed interval $[a, b]$, and k be any number such that $f(a), \leq k \leq f(b)$, then there exists $x, a \leq x \leq b$, such that:

$$f(x) = k$$

2.8 Self Assessment Exercises

(1.) Examine the continuity of the following function:

$$f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

at $x = 0$.

(2.) Examine the continuity of the following function at $x = 0, 1$

$$f(x) = \begin{cases} -x^2 & \text{if} \quad x \leq 0 \\ 5x - 4 & \text{if} \quad 0 < x \leq 1 \\ 4x^2 - 3x & \text{if} \quad x > 1 \end{cases}$$

Also discuss the type of discontinuity, if any.

(3.) Examine the continuity of the following function at $x = 0, 1/2, 3/4, 1$.

$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0, 1/2 \\ 1/2 - x & \text{if} \quad 0 < x \leq 1/2 \\ 4x^2 - 1 & \text{if} \quad 1/2 < x < 3/4 \\ 1 - x^2 & \text{if} \quad 3/4 \leq x \leq 1 \end{cases}$$

Also discuss the type of discontinuity, if any.

(4.) Show that the function defined as

$$f(x) = \begin{cases} \frac{x^2-a^2}{x-a} & \text{if} \quad x \neq a \\ a & \text{if} \quad x = a \end{cases}$$

is discontinuous at $x = a$.

(5.) Show that the function defined as

$$f(x) = \begin{cases} x + 1 & \text{if} \quad -1 \leq x < 0 \\ x & \text{if} \quad 0 \leq x \leq 1 \\ 2 - x & \text{if} \quad 1 < x \leq 2 \end{cases}$$

is discontinuous at $x = 0$ and continuous at $x = 1$.

(6.) Show that the function $f(x) = |x - 1| + |x - 2|$ is continuous at $x = 1$ and $x = 2$.

(7.) Show that the function $f(x) = x^5 - 2x^3 - 2 = 0$ has a root in the interval $(0, 2)$.

(8.) Use Intermediate Value Theorem to show the the equation $e^x = 4 - x^3$ is solvable.

(9.) Find $\lim_{x \to \sqrt{\pi}} cos(x^2)$.

(10.) Discuss the continuity of the function $f(x) = |3 + sin(2x)|$.

(11.) Discuss the continuity of the function $f(x) = sin(x^2 - 4)$.

(12.) Discuss the continuity of the following function $f(x) = \lfloor x \rfloor$ at $x = n, n \in \mathbf{Z}$, where $\lfloor x \rfloor$ denotes the greatest integer function defined as:

$$\lfloor x \rfloor = n \quad \text{when} \quad n \leq x < n + 1.$$

(13.) Examine the continuity of the function

$$f(x) = \begin{cases} \frac{x+|x|}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

(14.) Consider the following functions f and g, defined as:

$$f(x) = \begin{cases} \frac{2Sin(x)}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{Sin(x)}{x} & \text{if} \quad x \neq 0 \\ 2 & \text{if} \quad x = 0 \end{cases}$$

Show that $f + g$ is continuous at $x = 0$, whereas f and g are not continuous at $x = 0$.

(15.) Consider the following functions f and g, defined as:

$$f(x) = \begin{cases} \frac{2Sin(x)}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{Sin(x)}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Show that $f - g$ is continuous at $x = 0$, whereas f and g are not continuous at $x = 0$.

(16.) Consider the following functions f and g, defined as:

$$f(x) = \begin{cases} \frac{2Sin(x)}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

$$f(x) = \begin{cases} \frac{Sin(x)}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Show that $f - g$ is continuous at $x = 0$, whereas f and g are not continuous at $x = 0$.

(17.) Consider the following functions f, defined as:

$$f(x) = \begin{cases} \frac{x}{|x|} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

Show that $|f|$ is continuous at $x = 0$, though f, is not continuous at $x = 0$.

2.9 Solution of In-text Exercises

In-text Exercise 2.1

Solution. $\lim_{x \to 0^+} \dfrac{xe^{1/x}}{1 + e^{1/x}} = \lim_{x \to 0^+} \dfrac{x}{e^{-1/x} + 1} = \dfrac{0}{1} = 0.$

And $\lim_{x \to 0^-} \dfrac{xe^{1/x}}{1 + e^{1/x}} = \dfrac{0.0}{0 + 1} = 0.$

As $f(0) = 0$, f is continuous at $x = 0$.

In-text Exercise 2.2

Solution. Given that f is continuous at $x = 2$, implies $\lim_{x \to 2} f(x) = f(2) = a$.

Now $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1) = 3.$

and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + 1) = 3.$

this implies $\lim_{x \to 2} f(x) = 3$

$\implies a = f(2) = 3.$

In-text Exercise 2.3

Solution. Let $f(x) = x^4 + 3x^2 - 1$ and $g(x) = |x|$, then the given function is a composition of these two functions, that is $gof(x) = g(f(x)) = g(x^4 + 3x^2 - 1) = |x^4 + 3x^2 - 1|$.

Since the function $f(x) = x^4 + 3x^2 - 1$ and $g(x) = |x|$ are continuous everywhere, the composition function $gof(x)$ is continuous everywhere, by Result 2.2.

In-text Exercise 2.4

Solution. Since $tan(x)$ is continuous in the domain $\mathbb{R} - \{(2n + 1)\frac{\pi}{2} : n \in \mathbb{Z}\}$, we have

$$\lim_{x \to \pi} tan(2x) = tan(\lim_{x \to \pi} 2x)$$
$$= tan(2\pi) = 0 \text{ (Note } 2\pi \text{ belongs to the natural domain of } tan(x)).$$

2.10 Suggested Readings

1. Anton, Howard, Bivens, Irl, Davis, Stephen (2013). Calculus (10th ed.). Wiley India Pvt. Ltd. New Delhi. International Student Version. Indian Reprint 2016.

2. Prasad, Gorakh (2016). Differential Calculus (19th ed.). Pothishala Pvt. Ltd. Allahabad

3. Thomas Jr., George B., Weir, Maurice D., Hass, Joel (2014). Thomas' Calculus (13thed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 3

Differentiation

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Structure

3.1 Learning Objectives

1. To Understand the relationship between the rate of change and the derivative of a function.

2. To learn properties of a differentiable function.

3. To use the techniques of differentiation for finding the derivative of various functions.

3.2 Introduction

The concept of the rate of change is very much applicable in various fields of life, like the rate of change of speed of a rocket, bacteria growth in a culture, populating change of a species, pollution level in a lake, and many more. The concept of derivative is the mathematical tool to capture the rate of change, which is very much related to the concept of the tangent line to a curve. So let us first see how to derive the slope of a tangent to a curve using the concept of limits.

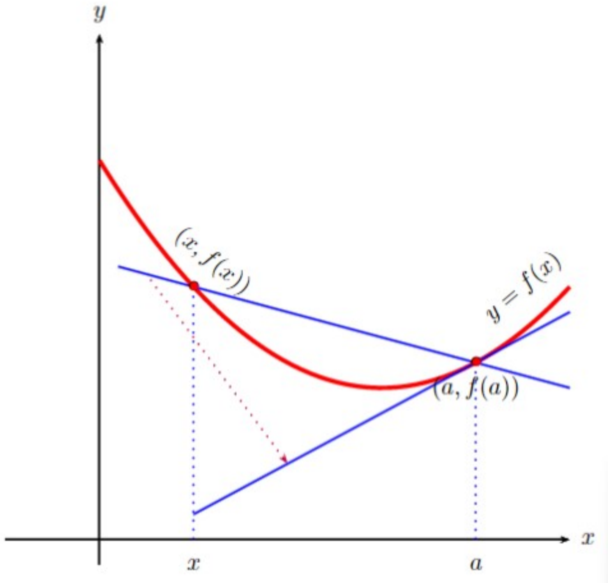

Figure 3.1: Tangent line to the curve

For example, consider the graph of the function shown in the Figure 3.1. Suppose we are interested in finding the slope of the tangent line at the point $(a, f(a))$. Take any other point $(x, f(x))$ on the curve. Let m be the slope of the secant line joining these two points. Then

$$m = \frac{f(x) - f(a)}{x - a}$$

Note that intuitively m is nothing but the average rate of change in $[a, x]$, which is the ratio of change in the function values (that is $f(x) - f(a)$) and change in the value of x (that is $(x - a)$). If we let the point x to move along the curve towards the point a that is $x \to a$, then this secant line will become the tangent line to the curve at the point $(a, f(a))$ and m will approach to the slope of the tangent line at $(a, f(a))$. That is, the slope m of the tangent line at $(a, f(a))$ is given by :

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

and equation of the tangent line at $(a, f(a))$ is given by:

$$y - f(a) = m(x - a).$$

The above formula for slope m can be modified to a much usual known formula by putting $x = a + h$. This will imply when $x \to a$, then $h \to 0$, so the updated formula for m becomes:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The above formula for rate of change is directly related to the derivative of a function at a point a, which we will study more now.

3.3 Derivative of a Function

Definition 3.1. (Derivative of a Function)

The derivative of a function f at a point x denoted as $f'(x)$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' is the set of all the points of the domain of f for which above limit exists.

$f'(x)$ is also denoted as $\frac{d}{dx} f(x)$.

If $f'(a)$ exists, then we say f is differentiable (derivable) at $x = a$.

Example 3.1. Find the derivative of the function $f(x) = x^3$ and use it to find the equation of the tangent at the point $x = 2$.

Solution. By the definition of the derivative, we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{x^3 + h^3 + 3x^2 h + 3xh^2 - x^3}{h} \\ &= \lim_{h \to 0} \frac{h^3 + 3x^2 h + 3xh^2}{h} \\ &= \lim_{h \to 0} (h^2 + 3x^2 + 3xh) \\ &= 3x^2 \end{aligned}$$

Thus, $f'(x) = 3x^2$ for all $x \in \mathbb{R}$

This gives, $f'(2) = 3.2^2 = 12$, therefore equation of tangent at $(2, f(2)) = (2, 8)$ is given by:

$$y - 8 = 12(x - 2).$$

We can apply the concept of derivative to find out velocity function $v(t)$ of a particle whose position function is given by the displacement function $s(t)$, where t denotes the time. The velocity $v(t)$ at time t is defined as:

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h}$$

Example 3.2. Let the position function of a moving particle is given by $s(t) = t^2 - 2t + 5$, where t represents the unit of time. Find the velocity function.

Solution. By the definition, the velocity function $v(t)$ is given by:

$$\begin{aligned}
v(t) &= \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} \\
&= \lim_{h\to 0} \frac{(t+h)^2 - 2(t+h) + 5 - (t^2 - 2t + 5)}{h} \\
&= \lim_{h\to 0} \frac{t^2 + h^2 + 2th - 2t - 2h + 5 - t^2 + 2t - 5}{h} \\
&= \lim_{h\to 0} \frac{h^2 + 2th - 2h}{h} \\
&= \lim_{h\to 0} (h + 2t - 2) \\
&= 2t - 2
\end{aligned}$$

Example 3.3. Show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

Solution. To examine, if the given function is differentiable at $x = 0$, we need to check if $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ exists or not.

Given, $f(x) = |x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$

Then

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} \\
&= \lim_{h\to 0^-} \frac{-h - 0}{h} \qquad (|h| = -h \,\text{for}\, h < 0) \\
&= \lim_{h\to 0^-} \frac{-h}{h} = -1
\end{aligned}$$

and

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} \\
&= \lim_{h\to 0^+} \frac{h - 0}{h}, \qquad (as\, |h| = h \,\text{for}\, h > 0) \\
&= \lim_{h\to 0^+} \frac{h}{h} = 1
\end{aligned}$$

Since the left-hand limit and the right hand limit are not equal, this implies that $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ does not exist, that is f is not differentiable at $x = 0$.

Definition 3.2. (**Differentiablity in an Interval**)

We say that f is differentiable in an open interval (a, b) if $f'(x)$ exists for all $x \in (a, b)$.

Similarly, the defintion holds for the open intervals of the type $(-\infty, b)$, (a, ∞), $(-\infty, \infty)$. If f is differentiable on $(-\infty, \infty)$, we say f is differentiable everywhere.

Theorem 3.1. *Every differentiable function is continuous, that is if a function f is differentiable at a, then it is continuous at a.*

Proof. Let the function f is differentiable at $x = a$, then by definition $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

To prove that f is continuous at $x = a$, we need to show $\lim_{x \to a} f(x) = f(a)$.

Consider

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a), x \neq a$$

Taking limit $x \to a$ on both sides, we have

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}(\frac{f(x) - f(a)}{x - a}(x - a))$$

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a}(x - a)$$

$$= f'(a).(a - a) = f'(a).0 = 0$$

$$\implies \lim_{x \to a} f(x) = f(a)$$

Therefore, f is continuous at $x = a$.

□

Note. The converse of the above theorem is not true. That is every continuous function is not differentiable. For example the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.

In-text Exercise 3.1. Show that the function $f(x) = x|x|$ is derivable at $x = 0$.

Example 3.4. Check the differentiability of the function

$$f(x) = \begin{cases} 2x - 3 & \text{if} \quad 0 \leq x \leq 2 \\ x^2 - 3 & \text{if} \quad 2 < x \leq 4 \end{cases}$$

at $x = 2$.

Solution. To check $f'(2)$ we need to check the existence of $\lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$.

We have

$$\lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^-} \frac{2(2 + h) - 3 - 1}{h}$$

$$= \lim_{h \to 0^-} \frac{2h}{h}$$

$$= \lim_{h \to 0^-} 2 = 2.$$

Similarly,

$$\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+}\frac{(2+h)^2-3-1}{h}$$
$$= \lim_{h\to 0^+}\frac{4+h^2+4h-4}{h}$$
$$= \lim_{h\to 0^+} h+4 = 4.$$

Since the left hand limit and the right hand limit are not equal, $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ does not exist. Hence given function is not differentiable at $x = 2$.

Example 3.5. Show that the function defined as

$$f(x) = \begin{cases} xsin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is not differentiable at $x = 0$.

Solution. We have,

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\,sin(1/h)}{h}$$
$$= \lim_{h\to 0} sin(1/h)$$

which does not exist.
This implies that the given function is not differentiable at $x = 0$.

Example 3.6. Show that the function defined as

$$f(x) = \begin{cases} x^2sin(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is not differentiable at $x = 0$.

Solution. We have,

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{(h^2\,sin(1/h)}{h}$$
$$= \lim_{h\to 0} h\,sin(1/h) = 0$$

This implies that the given function is differentiable at $x = 0$ and $f'(0) = 0$.

3.3.1 Algebra of Derivatives

Let f and g be differentiable functions at x and c be any constant, then

1. $\frac{d}{dx}(c.f(x)) = c.\frac{d}{dx}f(x)$.
2. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
3. $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$.
4. $\frac{d}{dx}f(x).g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$.
5. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}$.

3.3.2 Derivative of some standard functions

Polynomial Functions:

1. $\frac{d}{dx}c = 0$ where c is any real constant.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}x^n = n.x^{n-1}$, where n is any real number.
4. $\frac{d}{dx}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = (a_1 + 2a_2x + 3a_3x^2 + \ldots na_nx^{n-1})$.
   Trigonometric Functions:
5. $\frac{d}{dx}sin(x) = cos(x)$.
6. $\frac{d}{dx}cos(x) = -Sin(x)$.
7. $\frac{d}{dx}tan(x) = sec^2(x)$.
8. $\frac{d}{dx}sec(x) = sec(x)tan(x)$.
9. $\frac{d}{dx}cot(x) = -csc^2(x)$.
10. $\frac{d}{dx}csc(x) = -csc(x)cot(x)$.
   Inverse Trigonometric Functions:

11. $\dfrac{d}{dx}sin^{-1}(x) = \dfrac{1}{\sqrt{1-x^2}}$.

12. $\dfrac{d}{dx}cos^{-1}(x) = -\dfrac{1}{\sqrt{1-x^2}}$.

13. $\dfrac{d}{dx}tan^{-1}(x) = \dfrac{1}{1+x^2}$.

14. $\dfrac{d}{dx}cot^{-1}(x) = -\dfrac{1}{1+x^2}$.

15. $\dfrac{d}{dx}sec^{-1}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$.

16. $\dfrac{d}{dx}sec^{-1}(x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$.

Exponential Functions:

17. $\dfrac{d}{dx}a^x = a^x ln(x)$, where $a > 0$.

18. $\dfrac{d}{dx}e^{a.x} = a.e^{a.x}$.

Logarithmic Functions:

19. $\dfrac{d}{dx}ln(x) = \dfrac{1}{x}, x > 0$.

20. $\dfrac{d}{dx}log_b(x) = \dfrac{1}{x.ln(b)}, x > 0$.

Example 3.7. 1. $\frac{d}{dx}(x^\pi) = \pi x^{\pi-1}$.

2. $\frac{d}{dx}x^{5/4} = (5/4)x^{5/4-1} = (5/4)x^{1/4}$.

3. $\frac{d}{dx}(x^4 + x^{-9}) = \frac{d}{dx}x^4 + \frac{d}{dx}x^{-9} = 4x^3 - 9x^{-10}$.

4. $\frac{d}{dx}(5/x) = 5\frac{d}{dx}\frac{1}{x} = 5.\frac{-1}{x^2}$.

In-text Exercise 3.2. Find the derivative of $y = \dfrac{x^2-1}{x^2+1}$.

Example 3.8. At what points, the graph of the function $f(x) = 6x^3 - 18x + 1$ have the horizontal tangent line?

Solution. For the given function $f(x) = 6x^3 - 18x + 1$, the points at which graph has horizontal tangent is given by $f'(x) = 0$, that is $18x^2 - 18 = 0$, which gives $x = 1, -1$.

So the graph of the function has horizontal tangent line at the points $(1, -11)$ and $(-1, 13)$.

Example 3.9. Find $\frac{d}{dx}((x^2-5)(5x^3+x+1))$.

Solution.
$$\begin{aligned}\tfrac{d}{dx}((x^2-5)(5x^3+x+1)) &= (x^2-5)\tfrac{d}{dx}(5x^3+x+1)+(5x^3+x+1)\tfrac{d}{dx}(x^2-5)\\ &= (x^2-5)(15x^2+1)+(5x^3+x+1)(2x)\\ &= (15x^4-75x^2+x^2-5)+(10x^4+2x^2+2x)\\ &= 25^4-72x^2+2x-5.\end{aligned}$$

Example 3.10. Find $\frac{d}{dx}(\frac{x^2+1}{x+1})$.

Solution.
$$\begin{aligned}\frac{d}{dx}(\frac{x^2+1}{x+1}) &= \frac{(x+1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x+1)}{(x+1)^2}\\ &= \frac{(x+1)2x-(x^2+1).1}{(x+1)^2}\\ &= \frac{2x^2+2x-x^2-1}{(x+1)^2} = \frac{x^2+2x-1}{(x+1)^2}\end{aligned}$$

Example 3.11. Find $\frac{d}{dx}sin(x).(1+cos(x))$

Solution.
$$\begin{aligned}\tfrac{d}{dx}sin(x).(1+cos(x)) &= sin(x)\tfrac{d}{dx}(1+cos(x))+(1+cos(x))\tfrac{d}{dx}sin(x)\\ &= sin(x).-sin(x)+(1+cos(x))cos(x)\\ &= -sin^2(x)+cos(x)+cos^2(x).\end{aligned}$$

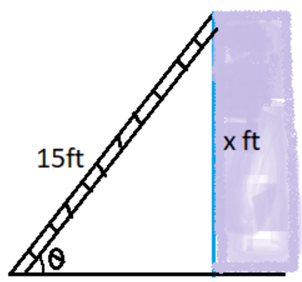

Figure 3.2: Ladder leaning against the wall.

Example 3.12. A 15 ft ladder makes an angle θ with the horizontal (ground) as shown in Figure 3.2. If the top of the ladder is x ft above the ground, Find the rate at which x changes with respect to θ, when $\theta = 60^0$. Convert the answer in units of feet/degree.

Solution. Here we have $sin(\theta) = x/15$, that is

$x = 15sin(\theta)$.

$\implies \frac{dx}{d\theta} = 15cos(\theta)$.

which gives the rate of change in x, with respect to angle θ in feet/radian.

When $\theta = 60^o$ or $\pi/3$ radians, we have

$\frac{dx}{d\theta} = 15cos(\pi/3) = 15.(1/2)$ (feet/radian).

Converting radian to degree, we get

$\frac{15}{2}\frac{\pi}{180} = 0.13$ (feet/degree).

3.4 Chain Rule

Theorem 3.2. ***Chain Rule***
If $y = f(u)$ and $u = g(x)$ are two derivable functions then,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$$

Alternatively, the above result can be stated as:
If g is differentiable at x and f is differentiable at $g(x)$, then the composition function $fog(x)$ is differentiable at x, and

$$\frac{d}{dx}f(g(x)) = (fog)'(x) = f'(g(x)).g'(x).$$

Example 3.13. Find $\frac{dy}{dx}$, where $y = sin(x^4)$.

Solution. Given $y = sin(x^4)$.
Let $u = x^4$, then $y = sin(u)$.
By the Chain Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{d}{du}sin(u).\frac{d}{dx}x^4 = cos(u).4x^3 = 4x^3cos(x^4).$$

Example 3.14. Find $\frac{dy}{dx}$, where $y = ln(x^2)$.

Solution. Given $y = ln(x^2)$.
Let $u = x^2$, then $y = ln(u)$.
By the Chain Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{d}{du}ln(u).\frac{d}{dx}x^2 = (1/u).2x = \frac{2x}{x^2} = \frac{2}{x}.$$

In-text Exercise 3.3. Find $\frac{dy}{dx}$, where $y = tan^{-1}(\sqrt{x})$.

3.4.1 Derivative of Parametric Function

Theorem 3.3. *If $x = f(t)$ and $y = g(t)$ are two differentiable functions of t, then*

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad dx/dt \neq 0.$$

Example 3.15. if $x(t) = a(cos(4t) - sin(2t))$ and $y(t) = e^{2t}$, Find dy/dx.

Solution. Here x and y are functions of parameter t, therefore

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{a(-4sin(4t) - 2cos(2t))}{2e^{2t}} = \frac{-a(2sin(4t) + cos(2t)))}{e^{2t}}.$$

Example 3.16. If $x(t) = cos^2(t)$ and $y(t) = sin^3(t)$, Find dy/dx.

Solution. Here x and y are functions of parameter t, therefore

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$= \frac{2cos(t)(-sin(t))}{3sin^2(t).cos(t)} = \frac{-2}{3sin(t)} = \frac{-2}{3}csc(t).$$

In-text Exercise 3.4. If $x(t) = a(cos(t) + ln(tan(t/2))$ and $y(t) = asin(t)$, Find dy/dx.

3.5 Summary

1. The derivative of a function f at a point x denoted as $f'(x)$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Every differentiable function is continuous.

3. A function $f(x)$ is differentiable in an open interval (a, b) if $f'(x)$ exists for all $x \in (a, b)$.

4. Algebra of Derivatives: If f and g are differentiable functions at x and c be any constant, then

   (a) $\frac{d}{dx}(c.f(x)) = c.\frac{d}{dx}f(x)$.

   (b) $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.

   (c) $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$.

   (d) $\frac{d}{dx}f(x).g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$.

   (e) $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}$.

5. Chain Rule: if $y = f(u)$ and $u = g(x)$ are two derivable functions then,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$$

6. If $x = f(t)$ and $y = g(t)$ are two differentiable functions of t, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad dx/dt \neq 0.$$

3.6 Self Assessment Exercises

1. Show that the function $f(x) = |x - 1|$ is not differentiable at $x = 1$.

2. Show that the function defined as

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 1 \\ x & \text{if} \quad x \geq 1 \end{cases}$$

   is continuous at $x = 1$, but not differentiable at $x = 1$.

3. Show that the function $f(x) = |x - 1| + |x + 1|$ is not differentiable at $x = -1, 1$.

4. Show that the function $f(x)$ defined as

$$f(x) = \begin{cases} xtan^{-1}(x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

   is not differentiable at $x = 0$.

5. Given an example of a function which is continuous at $x = 1$, but not differentiable at $x = 1$.

6. If $y = x^{3/2} + x$, find $\frac{dy}{dx}$ at $x = 1$.

7. Find $\frac{ds}{dt}$, if $s(t) = (1 + t^2)\sqrt{t}$.

8. Find $\frac{dy}{dx}$, if $y(x) = \frac{1+x^2}{x^3}$.

9. Find $\frac{dy}{dx}$, if $y(x) = (2x^2 + 1)(1 + \frac{1}{x^2})x^{5/2}$.

10. Find $\frac{dy}{dx}$, if $y(x) = sec^2(x) + tan^(x)$.

11. Find $\frac{dy}{dx}$, if $y(x) = \frac{sin^{-1}(x)}{1+tan(x)}$.

12. Find $\frac{dy}{dx}$, if $y(x) = (\frac{1+x^2}{1-x^2})^5$.

13. Find $\frac{dy}{dx}$, if $y(x) = ln(\frac{x}{a+x^2})$.

14. Find $\frac{dy}{dx}$, if $y(x) = sin^2(ln(x))$.

15. Find $\frac{dy}{dx}$, if $y(x) = ln(cos^{-1}(x))$.

16. Find $\frac{dy}{dx}$, if $x(t) = e^t(cos(t) - sin(t)), y(t) = e^t(cos(t) + sin(t))$.

3.7 Solution of In-text Exercises

In-text Exercise 3.1

Solution. To show that $f'(0)$ exists we need to show that the limit $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists.

Let's find the left-hand limit first.

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^-} \frac{h|h|-0|0|}{h} \\
&= \lim_{h\to 0^-} \frac{h|h|}{h} \\
&= \lim_{h\to 0^-} |h| \\
&= \lim_{h\to 0^-} (-h) = 0
\end{aligned}$$

Let's find the right-hand limit now.

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{h|h|-0|0|}{h} \\
&= \lim_{h\to 0^+} \frac{h|h|}{h} \\
&= \lim_{h\to 0^+} |h| \\
&= \lim_{h\to 0^+} h = 0
\end{aligned}$$

Since, the left-hand limit and the right-hand limit are equal, the limit $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h}$ exists. Hence given function is differentiable at $x = 0$.

In-text Exercise 3.2

Solution. $\frac{dy}{dx} = \frac{x^2-1}{x^2+1} = \frac{(x^2+1)2x-(x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$.

In-text Exercise 3.3

Solution. Given $y = tan^{-1}(\sqrt{x})$.

Let $u = \sqrt{x}$, then $y = tan^{-1}(u)$.

By Chain Rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}.\frac{du}{dx} \\
&= \frac{d}{du}tan^{-1}(u).\frac{d}{dx}\sqrt{x} \\
&= \frac{1}{1+u^2}.\frac{1}{2\sqrt{x}} = \frac{1}{1+x}.\frac{1}{2\sqrt{x}}.
\end{aligned}$$

In-text Exercise 3.4

Solution. $= \dfrac{dx}{dt} = a(-sin(t) + \dfrac{sec^2(t/2)}{tan(t/2)}\dfrac{1}{2}) = a(-sin(t) + \dfrac{1}{sin(t)}) = \dfrac{acos^2(t)}{sin(t)}$

and $\dfrac{dy}{dt} = acos(t)$

Hence, $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = tan(t)$.

3.8 Suggested Readings

1. Anton, Howard, Bivens, Irl, Davis, Stephen (2013). Calculus (10th ed.). Wiley India Pvt. Ltd. New Delhi. International Student Version. Indian Reprint 2016.

2. Prasad, Gorakh (2016). Differential Calculus (19th ed.). Pothishala Pvt. Ltd. Allahabad

3. Thomas Jr., George B., Weir, Maurice D., Hass, Joel (2014). Thomas' Calculus (13thed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 4

Successive Differentiation and Partial Differentiation

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Structure

4.1 Learning Objectives

1. To understand the concept of successive differentiation.
2. To apply Leibinitz's theorem to find out the n^{th} derivative of the product of two functions.
3. To understand the concept of partial differentiation.
4. Understanding Euler's theorem for homogeneous functions.
5. To learn applications of Euler's theorem.

4.2 Introduction

In the previous lesson, we have learned about the concept of differentiability. The derivative $f'(x)$ of a function $f(x)$ is either a constant or a function of x, if it exists. Therefore, $f'(x)$ may further be examined for differentiability. In this lesson, we will study the higher-order derivatives of some standard functions and apply Leibinitz‘s theorem to find out higher-order derivatives for the product of two functions.

Partial differentiation deals with the differentiation of a function of two or more variables. For example, the volume of a right circular cylinder depends on its radius and height, the area of a triangle depends on the base length and its height. In this lesson, we will study the concept of partial derivatives for functions with more than one variable. Further, we will study about homogenous functions and Euler's theorem for such functions.

4.3 Higher Order Derivatives

Let $y = f(x)$ be a function differentiable at of $x \in \mathbb{R}$.

The first derivative of $y = f(x)$ if exist, is denoted as $y_1 = \dfrac{dy}{dx} = f'(x)$.

The second derivative of $y = f(x)$ is denoted as $y_2 = \dfrac{dy_1}{dx} = \dfrac{d}{dx}(\dfrac{dy}{dx}) = \dfrac{d^2y}{dx^2} = f''(x)$.

Similarly, the third derivative is defined as $y_3 = \dfrac{dy_2}{dx} = \dfrac{d}{dx}(\dfrac{d^2y}{dx^2}) = \dfrac{d^3y}{dx^3} = f'''(x)$, and so on.

The n^{th} derivative is defined as $y_n = \dfrac{dy_{n-1}}{dx} = \dfrac{d}{dx}(\dfrac{d^{n-1}y}{dx^{n-1}}) = \dfrac{d^ny}{dx^n} = f^n(x)$.

Example 4.1. If $x = a(cos(\theta) + \theta sin(\theta)), y = a(sin(\theta) - \theta cos(\theta))$, find $\frac{d^2y}{dx^2}$.

Solution. Differentiating $x = a(cos(\theta) + \theta sin(\theta))$, and
$y = a(sin(\theta) - \theta cos(\theta))$, with respect to θ, we obtain

$$\frac{dx}{d\theta} = a(-sin(\theta) + \theta cos(\theta) + sin(\theta)) = a\theta cos(\theta)$$

$$\frac{dy}{d\theta} = a(cos(\theta) + \theta sin(\theta) - cos(\theta)) = a\theta sin(\theta)$$

$$\implies \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = tan(\theta)$$

$$\implies \frac{d^2y}{dx^2} = \frac{d}{d\theta}(tan(\theta)).\frac{d\theta}{dx} = sec^2(\theta).\frac{1}{a.\theta.cos(\theta)} = \frac{sec^3(\theta)}{a\theta}.$$

Example 4.2. If $y = sin(sin(x))$, then show that:

$$y_2 + tan(x)y_1 + cos^2(x)y = 0.$$

where, $y_1 = \dfrac{dy}{dx}, y_2 = \frac{d^2y}{dx^2}$.

Solution. We have $y = sin(sin(x))$
$\implies y_1 = cos(sin(x))cos(x) \quad \ldots(1)$
$\implies y_2 = -sin(sin(x))cos(x)cos(x) - cos(sin(x))sin(x)$
$\implies y_2 = -ycos^2(x) - y_1 sin(x)/cos(x) \quad (using(1))$
$\implies y_2 + tan(x)y_1 + cos^2(x)y = 0$

Example 4.3. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, then show that

$$\frac{d^2y}{dx^2} = -\frac{a}{(1-x^2)^{3/2}}$$

Solution. We have $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$
Substituting $x = sin(\theta), y = sin(\phi)$, we have
$sin(\theta)cos(\phi) + sin(\phi)cos(\theta) = a$
$\implies sin(\theta + \phi) = a \implies \theta + \phi = sin^{-1}(a)$
$\implies sin^{-1}(x) + sin^{-1}(y) = sin^{-1}(a)$
Differentiating both sides with respect to x, we obtain
$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$
$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \quad \ldots(1)$
Differentiating again with respect to x, we obtain
$\frac{d^2y}{dx^2} = \frac{\sqrt{1-x^2}\frac{-y}{\sqrt{1-y^2}}\frac{dy}{dx} - \sqrt{1-y^2}\frac{-x}{\sqrt{1-x^2}}}{1-x^2}$
$\implies \frac{d^2y}{dx^2} = \frac{-1}{1-x^2}\left(y + x\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right) \quad \ldots using(1)$
$\implies \frac{d^2y}{dx^2} = \frac{-1}{(1-x^2)^{3/2}}[x\sqrt{1-y^2} + y\sqrt{1-x^2}] = \frac{-a}{(1-x^2)^{3/2}}$ (by using the given relation.)

In-text Exercise 4.1. If $x = sin(t), y = sin(at)$ then show that:

$$(1-x^2)y_2 - xy_1 + a^2y = 0$$

4.3.1 n^{th} Order Derivatives of some Standard Functions

(a) $y = \frac{1}{ax+b}$, where $a, b \in \mathbb{R}$.

$$\begin{aligned}\implies y_1 &= -1.a(ax+b)^{-2}\\ y_2 &= -1.-2.a^2(ax+b)^{-3} = (-1)^2.2!.a^2(ax+b)^{-3}\\ y_3 &= (-1)^2.2!.(-3).a^3(ax+b)^{-4} = (-1)^3.3!.a^3(ax+b)^{-4}\\ &\vdots\\ y_n &= (-1)^n n!.a^n(ax+n)^{-(n+1)}\end{aligned}$$

(b) $y = (ax+b)^m, m \in \mathbb{N}$, a and b are real constants.

$$\begin{aligned}
\implies y_1 &= m.a(ax+b)^{m-1}\\
y_2 &= m(m-1).a^2(ax+b)^{m-2}\\
y_3 &= m(m-1)(m-2)a^3(ax+b)^{m-3}\\
&\vdots\\
y_n &= m(m-1)(m-2)\ldots(m-(n-1))a^n(ax+n)^{m-n} = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}.
\end{aligned}$$

(c) $y = \log(ax+b)$

$$\begin{aligned}
y_1 &= \frac{a}{ax+b}\\
y_n &= (n-1)th \text{ derivative of } \frac{a}{ax+b}\\
&= \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n} \quad \text{by using part (a).}
\end{aligned}$$

(d) $y = sin(ax+b)$

$$\begin{aligned}
y_1 &= acos(ax+b) = asin(ax+b+\frac{\pi}{2})\\
y_2 &= a^2cos(ax+b+\frac{\pi}{2}) = a^2sin(ax+b+\frac{2\pi}{2})\\
y_3 &= a^3cos(ax+b+\frac{2\pi}{2}) = a^3sin(ax+b+\frac{3\pi}{2})\\
&\vdots\\
y_n &= a^nsin(ax+b+\frac{n\pi}{2})
\end{aligned}$$

(e) $y = cos(ax+b)$

Following the process of part (d), we obtain

$$y_n = a^ncos(ax+b+\frac{n\pi}{2})$$

(f) $y = e^{ax}sin(bx+c)$

$$
\begin{aligned}
y_1 &= ae^{ax}sin(bx+c) + be^{ax}cos(bx+c)\\
&= e^{ax}(asin(bx+c) + bcos(bx+c))\\
&\text{Substituting } a = rcos(\theta), b = rsin(\theta)\\
&\implies r = \sqrt{a^2+b^2}, tan(\theta) = b/a\\
\implies y_1 &= re^{ax}(sin(bx+c)cos(\theta) + cos(bx+c)sin(\theta))\\
&= re^{ax}sin(bx+c+\theta)\\
\text{Similarly, } y_2 &= r^2e^{ax}sin(bx+c+2\theta)\\
&\vdots\\
y_n &= r^ne^{ax}sin(bx+c+n\theta), \text{ where } r = \sqrt{a^2+b^2}, \theta = tan^{-1}(b/a).
\end{aligned}
$$

(g) $y = e^{ax}cos(bx+c)$

Following the process of part (f), we obtain

$$
y_n = r^ne^{ax}cos(bx+c+n\theta), \text{ where } r = \sqrt{a^2+b^2}, \theta = tan^{-1}(b/a).
$$

Example 4.4. Find the n^{th} derivative of $\dfrac{1}{1-7x+12x^2}$.

Solution. Let's convert the given function into some standard known function, whose n^{th} derivative formula is known to us.

Using the partial fractions we have:

$$
\begin{aligned}
y &= \frac{1}{1-7x+12x^2} = \frac{1}{(1-3x)(1-4x)}\\
&= \frac{4}{1-4x} - \frac{3}{1-3x}\\
\implies y_n &= 4\frac{(-4)^n(-1)^nn!}{(1-4x)^{n+1}} - 3\frac{(-3)^n(-1)^nn!}{(1-3x)^{n+1}} \quad \text{by using 4.3.1(a)}\\
&= n!\left(\left(\frac{4}{1-4x}\right)^{n+1} - \left(\frac{3}{1-3x}\right)^{n+1}\right)
\end{aligned}
$$

Example 4.5. Find the n^{th} derivative of $y = sin(3x)cos(2x)$.

Solution. Given $y = sin(3x)cos(2x)$

$$
\begin{aligned}
&= \tfrac{1}{2}(sin(5x) + sin(x))\\
\implies y_n &= \tfrac{1}{2}(5^nsin(5x+n\pi/2) + 1^nsin(x+n\pi/2)) \text{ using Result 4.3.1(d, e).}
\end{aligned}
$$

Example 4.6. Fin the n^{th} derivative of $y = e^{2x}sin^4(x)$.

$y = e^{2x} sin^4(x)$
$= e^{2x}(1/4)(2sin^2(x))^2$
$= e^{2x}(1 - cos(2x))^2$

$$\implies y = \frac{e^{2x}}{4}(1 + cos^2(2x) - 2cos(2x))$$
$$= e^{2x}(\frac{1}{4} + \frac{1}{8}(2cos(2x))^2 - \frac{cos(2x)}{2})$$
$$= e^{2x}(\frac{1}{4} + \frac{1}{8} + \frac{1}{8}cos(4x) - \frac{cos(2x)}{2})$$
$$= \frac{3e^{2x}}{8} + \frac{1}{8}e^{2x}cos(4x) - \frac{1}{2}e^{2x}cos(2x)$$
$$\implies y_n = \frac{3.2^n}{8}e^{2x} + \frac{1.r^n}{8}e^{2x}cos(4x + n\theta) - \frac{1.s^n}{2}e^{2x}cos(2x + n\psi),$$ using 4.3.1(d, e).

where $r = \sqrt{2^2 + 4^2}, \theta = tan^{-1}(4/2), s = \sqrt{2^2 + 2^2}, \psi = \tan^{-1}(2/2)$.

In-text Exercise 4.2. If $y = xlog(\frac{x-1}{x+1})$, show that

$$y_n = (-1)^n(n-2)!\left(\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n)}\right)$$

4.4 n^{th} Derivative of Product of two Functions

Theorem 4.1. ***(Leibnitz's Theorem)***
If $u(x)$ and $v(x)$ are any two functions having derivative up to n^{th} order, then we have:

$$(u.v)_n = u_n.v + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + \cdots + \binom{n}{r}u_{n-r}v_r + \cdots + \binom{n}{n-1}u_1v_{n-1} + \binom{n}{n}uv_n$$

Here f_n denotes the n^{th} derivative of function f.

Example 4.7. If $y = (sin^{-1}(x))^2$, show that:

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0.$$

Solution. Given $y = (sin^{-1}(x))^2 \qquad \ldots (1)$

$\implies y_1 = \frac{2sin^{-1}(x)}{\sqrt{1-x^2}}$.

Squaring on both sides, and using (1), we have:
$(1 - x^2)y_1^2 = 4y$ using (1)
$\implies (1 - x^2)y_1^2 - 4y = 0$
Differentiating again w.r.t. x, we get
$(1 - x^2)2y_1y_2 - 2xy_1^2 - 4y_1 = 0$

$\implies (1-x^2)y_2 - xy_1 - 2 = 0 \qquad \ldots(3)$

Differentiating (3), n times using Leibnitz's Theorem, we have

Taking$u = y_2, v = (1-x^2)$ in $(1-x^2)y_2$ and $u = y_1, v = x$ in xy_1.

$$\left[(1-x^2)y_{n+2} - \binom{n}{1}2xy_{n+1} - \binom{n}{2}2y_n\right] - \left[xy_{n+1} + \binom{n}{1}y_n\right] = 0$$

$\implies (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n = 0$

$\implies (1-x^2)y_{n+2} - x(2n+1)y_{n+1} - n^2y_n = 0$ (Hence Proved.)

Example 4.8. If $y = [x + \sqrt{1+x^2}]^a$, then prove that:

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - a^2)y_n = 0$$

Solution. Given $y = [x + \sqrt{1+x^2}]^a \qquad \ldots(1)$

Differentiating w.r.t. x, we have

$y_1 = a[x + \sqrt{1+x^2}]^{a-1}\left[1 + \frac{2x}{2\sqrt{1+x^2}}\right]$

$\implies y_1 = \frac{ay}{\sqrt{1+x^2}}$ using (1)

$\implies \sqrt{1+x^2}y_1 = ay$

Squaring on both sides, we get

$(1+x^2)y_1^2 = a^2y^2$

Differentiating w.r.t. x, we have

$2(1+x^2)y_1y_2 + 2xy_1^2 - 2a^2yy_1 = 0$

$\implies (1+x^2)y_2 + xy_1 - a^2y = 0 \qquad \ldots(2)$ (cancelling $2y_1$)

Differentiating (2), n times using Leibnitz's rule, we have:

$$\left[(1+x^2)y_{n+2} + \binom{n}{1}2xy_{n+1} + \binom{n}{2}2y_n\right] + \left[xy_{n+1} + \binom{n}{1}y_n\right] - a^2y_n = 0$$

$\implies (1+x^2)y_{n+2} + 2xny_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - a^2y_n = 0$

$\implies (1+x^2)y_{n+2} + x(2n+1)y_{n+1} + (n^2-a^2)y_n = 0$. (Hence Proved)

Example 4.9. If $y = acos(log(x)) + bsin(log(x))$, then show that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Solution. Given $y = acos(log(x)) + bsin(log(x)) \qquad \ldots(1)$

Differentiating w.r.t. x, we get

$y_1 = -\frac{a}{x}sin(log(x)) + \frac{b}{x}cos(log(x))$

$\implies xy_1 = -asin(log(x)) + bcos(log(x))$

Differentiating again w.r.t. x, we have

$xy_2 + y_1 = -\frac{a}{x}cos(log(x)) - \frac{b}{x}sin(log(x))$

$\implies x^2y_2 + xy_1 = -acos(log(x)) - bsin(log(x)) = -y \quad$ using (1)

$\implies x^2y_2 + xy_1 + y = 0$

Now Differentiating n times, by using Leibnitz's theorem, we get

$$\left[x^2y_{n+2} + \binom{n}{1}2xy_{n+1} + \binom{n}{2}2y_n\right] + \left[xy_{n+1} + \binom{n}{1}y_n\right] + y_n = 0$$

$\implies x^2y_{n+2} + 2xny_{n+!} + n(n-1)y_n + xy_{n+1} + ny_n + y_n = 0$

$\implies x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0 \quad$ (Hence Proved)

Example 4.10. If $cos^{-1}\left(\frac{y}{b}\right) = log\left(\frac{x}{n}\right)^n$, then show that:

$$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0.$$

Solution. Given $cos^{-1}\left(\frac{y}{b}\right) = n(log(x) - log(n))$.
$\implies y = b\,cos(nlog(x) - nlog(n)) \qquad \ldots(1)$
Differentiating w.r.t. x, we have
$y_1 = -b\,sin(nlog(x) - nlog(n))(n/x)$
$\implies xy_1 = -nb\,sin(nlog(x) - nlog(n))$
Differentiating again w.r.t. x, we get
$xy_2 + y_1 = -nb\,cos(nlog(x) - nlog(n))(n/x)$
$\implies x^2y_2 + xy_1 = -n^2b\,cos(nlog(x) - nlog(n)) = -n^2y \quad$ using (1)
$\implies x^2y_2 + xy_1 + n^2y = 0$
Differentiating n times by using Leibnitz's theorem, we get:
$\left[x^2y_{n+2} + \binom{n}{1}2xy_{n+1} + \binom{n}{2}2y_n\right] + \left[xy_{n+1} + \binom{n}{1}y_n\right] + n^2y_n = 0$
$\implies x^2y_{n+2} + 2xny_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n + n^2y_n = 0$
$\implies x^2y_{n+2} + (2n+1)xy_{n+1} - n^2y_n = 0 \quad$ (Hence Proved)

Example 4.11. If $f(x) = tan(x)$, then show that:

$$f_n(0) - \binom{n}{2}f_{n-2}(0) + \binom{n}{4}f_{n-4}(0) - \cdots = sin(\frac{n\pi}{2}).$$

Solution. Given that $f(x) = tan(x) = \frac{sin(x)}{cos(x)}$
$\implies cos(x)f(x) = sin(x)$
Differentiating n times by using Leibnitz's Theorem
$\implies f_n(x)cos(x) + \binom{n}{1}f_{n-1}(x)(-sin(x)) + \binom{n}{2}f_{n-2}(x)(-cos(x)) + \binom{n}{3}f_{n-3}(x)sin(x) + \binom{n}{4}f_{n-4}(x)cos(x) + \cdots = sin(x + \frac{n\pi}{2})$
Putting $x = 0$ in above equation , we have
$f_n(0) - \binom{n}{2}f_{n-2}(0) + \binom{n}{4}f_{n-4}(0) - \cdots = sin(\frac{n\pi}{2})$.

Example 4.12. By forming the n^{th} derivative of x^{2n} in two different ways, show that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^22^2} + \frac{n^2(n-1)^2(n-2)^2}{1^22^23^2} + \cdots = \frac{(2n)!}{(n!)^2}.$$

Solution. One can easily see that

$$\frac{d}{dx^n}x^n = n!, \frac{d}{dx^{n-1}}x^n = n!x, \frac{d}{dx^{n-2}}x^n = \frac{n!}{2!}x^2, \frac{d}{dx^{n-3}}x^n = \frac{n!}{3!}x^3, \ldots$$

Now lets write $x^{2n} = x^n.x^n$

Differentiating both sides n time and applying Leibnitz's theorem to right hand side:

$$\begin{aligned}\frac{d}{dx}x^{2n} &= (n!)x^n + \binom{n}{1}(n!x)(nx^{n-1}) + \binom{n}{2}(\frac{n!}{2!}x^2)(n(n-1)x^2) \\ &+ \binom{n}{3}(\frac{n!}{3!}x^3)(n(n-1)(n-2)x^3) + \cdots + x^n(n!) \\ &= n!x^n\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \ldots\right] \quad \ldots(1)\end{aligned}$$

Now

$$\begin{aligned}\frac{d}{dx}x^{2n} &= 2n(2n-1)(2n-2)\ldots(2n-(n-1))x^{2n-n} \\ &= 2n(2n-1)(2n-2)\ldots(n+1)x^n \\ &= \frac{2n(2n-1)(2n-2)\ldots(n+1)n(n-1)\ldots 3.2.1x^n}{n(n-1)\ldots 3.2.1} \\ &= \frac{(2n)!x^n}{n!} \qquad \ldots(2)\end{aligned}$$

Comparing (1) and (2), we have

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 2^2 3^2} + \cdots = \frac{(2n)!}{(n!)^2} \quad (Hence Proved).$$

Example 4.13. If $y = e^{tan^{-1}(x)}$, show that

$$(1+x^2)y_{n+2} + (2(n+1)x-1)y_{n+1} + n(n+1)y_n = 0$$

Solution. If $y = e^{tan^{-1}(x)}$

Differentiating w.r.t. x, we get

$y_1 = e^{tan^{-1}(x)}\frac{1}{1+x^2}$

$\implies (1+x^2)y_1 = y$

Differentiating again w.r.t. x, we get

$\implies (1+x^2)y_2 + 2xy_1 = y_1$

$\implies (1+x^2)y_2 + 2xy_1 - y_1 = 0$

Differentiating n times, by using Leibnitz's Theorem, we get

$\implies (1+x^2)y_{n+2} + \binom{n}{1}2xy_{n+1} + \binom{n}{2}2y_n + 2xy_{n+1} + \binom{n}{1}2y_n - y_{n+1} = 0$

$\implies (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + 2xy_{n+1} + 2ny_n - y_{n+1} = 0$

$\implies (1+x^2)y_{n+2} + (2(n+1)x-1)y_{n+1} + n(n+1)y_n = 0$

Example 4.14. If $y = sin(msin^{-1}(x))$, show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Further, find out $y_n(0)$.

Solution. Given $y = sin(msin^{-1}(x)) \qquad \ldots(1)$
Differentiating w.r.t. x, we get
$y_1 = cos(m\,sin^{-1}(x))\dfrac{m}{\sqrt{1-x^2}} \qquad \ldots(2)$
$\implies (1-x^2)y_1^2 = m^2\,cos^2(m\,sin^{-1}(x))$
$= m^2(1 - sin^2(m\,sin^{-1}(x))) = m^2(1-y^2)$
$\implies (1-x^2)y_1^2 = m^2(1-y^2)$
Differentiating again w.r.t. x, we get
$\implies (1-x^2)2y_1y_2 - 2xy_1^2 = -2m^2yy_1$
$\implies (1-x^2)y_2 - xy_1 + m^2y = 0 \qquad \ldots(3)$ (cancelling $2y_1$ from both sides)
Differentiating n times, by using Libenitz's Theorem, we get
$(1-x^2)y_{n+2} + \binom{n}{1}(-2x)y_{n+1} + \binom{n}{2}(-2)y_n - xy_{n+1} - \binom{n}{1}y_n + m^2y_n = 0$
$\implies (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$
$\implies (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0)$
Now for $x = 0$, we get
$y_{n+2}(0) = (n^2-m^2)y_n(0) \qquad \ldots(4)$
Also $y(0) = 0$ from (1)
$y_1(0) = m$ from (2)
$y_2(0) = 0$ from (3)
Substituting $n = 1, 2, 3, \ldots$ in (4), we have
$y_3(0) = (n^2-m^2)y_1(0) = (1^2-m^2)m$
$y_4(0) = (n^2-m^2)y_2(0) = (2^2-m^2)0 = 0$
$y_5(0) = (n^2-m^2)y_3(0) = (3^2-m^2)(1^2-m^2)m$
Generalizing the above pattern we can conclude:

$$y_n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ m[(1^2-m^2)(3^2-m^2)\ldots((n-2)^2-m^2)] & \text{if } n \text{ is odd} \end{cases}$$

Example 4.15. If $y = log(x + \sqrt{1+x^2})$, show that

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$$

Hence, find $y_n(0)$.

Solution. Given $y = log(x + \sqrt{1+x^2}) \qquad \ldots(1)$
Differentiating w.r.t. x, we get
$y_1 = \dfrac{1}{x+\sqrt{1+x^2}}(1 + \dfrac{x}{\sqrt{1+x^2}})$
$\implies \sqrt{1+x^2}y_1 = 1$
$\implies (1+x^2)y_1^2 = 1 \qquad \ldots(2)$
Differentiating again w.r.t. x, we get

$(1+x^2)2y_1y_2 + 2xy_1^2 = 0$
$\implies (1+x^2)y_2 + xy_1 = 0 \qquad \ldots(3)$
Differentiating n times by using Leibnitz's Theorem, we get
$(1+x^2)y_{n+2} + \binom{n}{1}2xy_{n+1} + \binom{n}{2}2y_n + xy_{n+1} + \binom{n}{1}y_n = 0$
$\implies (1+x^2)y_{n+2} + 2xny_{n+!} + n(n-1)y_n + xy_{n+1} + ny_n = 0$
$\implies (1+x^2)y_{n+2} + x(2n+1)y_{n+1} + n^2y_n = 0 \qquad \ldots(4)$
Putting $x = 0$ in (4), we have
$y_{n+2}(0) = -n^2y_n(0) \qquad \ldots(5)$
Putting $x = 0$ in (1), (2) and (3), we get
$y(0) = 0, y_1(0) = 1, y_2(0) = 0.$
Now substituting $n = 1, 2, 3, \ldots$ in (4), we get
$y_3(0) = -1^2y_1(0) = -(1)^2$
$y_4(0) = -2^2y_2(0) = 0$
$y_5(0) = -3^2y_3(0) = -(1)^2.1^2.3^2$
$y_6(0) = -4^2y_4(0) = 0$
$y_7(0) = -5^2y_5(0) = -(1)^3.1^2.3^2.5^2$
Generalising the above pattern we get:
$y_n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^n1^23^35^2(n-1)^2 & \text{if } n \text{ is odd} \end{cases}$

In-text Exercise 4.3. If $y = tan^{-1}(x)$, prove that

$$(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$$

Hence, find $y_n(0)$

4.5 Partial Differentiation

Definition 4.1. Let $z = f(x, y)$ be a function of two variables x and y. The partial derivative of z w.r.t. x, denoted as $\frac{\partial z}{\partial x}$, is defined as:

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h},$$

provided the limit exists.

Note that y is treated as constant in the above definition.

Similarly, the partial derivative of z w.r.t. y, denoted as $\frac{\partial z}{\partial y}$ is defined as:

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

provided the limit exists.

Note that x is treated as constant in the above definition.

The partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called first-order partial derivatives of $z = f(x, y)$ with respect to x and y respectively. They are also denoted as $\frac{\partial f}{\partial x}$ or f_x and $\frac{\partial f}{\partial y}$ or f_y.

Result 4.1. 1. To find out $\frac{\partial z}{\partial x}$, differentiate $z = f(x, y)$, w.r.t. x, treating y as constant.

2. To find out $\frac{\partial z}{\partial y}$, differentiate $z = f(x, y)$, w.r.t. y, treating x as constant.

The functions $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are again functions of two variables, and we can further find out partial derivatives of these functions, which are defined as:

- $\frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) = \frac{\partial^2 z}{\partial x^2}$.
- $\frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) = \frac{\partial^2 z}{\partial y^2}$.
- $\frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) = \frac{\partial^2 z}{\partial x \partial y}$.
- $\frac{\partial}{\partial y}(\frac{\partial z}{\partial x}) = \frac{\partial^2 z}{\partial y \partial x}$.

These are called second order partial derivatives.

Note. One must note that in general $\frac{\partial^2 z}{\partial x \partial y} \neq \frac{\partial^2 z}{\partial y \partial x}$.

Theorem 4.2. *If the second order partial derivatives* $\frac{\partial^2 z}{\partial x \partial y}$ *and* $\frac{\partial^2 z}{\partial y \partial x}$ *are continuous, then*

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

Example 4.16. Find the first order and second order partial derivatives of

$$(i) z = x^3 + y^3 + 12xy \qquad (ii) z = (tan^{-1})(x/y)$$

Solution. (i)

Given $z = x^3 + y^3 + 12xy$

$\implies \frac{\partial z}{\partial x} = 3x^2 + 12y$ (Treating y as a constant)

$\implies \frac{\partial z}{\partial y} = 3y^2 + 12x$ (Treating x as a constant)

$\implies \frac{\partial^2 z}{\partial x^2} = 6x$

$\implies \frac{\partial^2 z}{\partial y^2} = 6y$

$\implies \frac{\partial^2 z}{\partial x \partial y} = 12$

$\implies \frac{\partial^2 z}{\partial y \partial x} = 12$

(Note that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$)

Solution. (ii)

Given $z = (tan^{-1})(x/y)$

$\implies \dfrac{\partial z}{\partial x} = \dfrac{1}{1+\frac{x^2}{y^2}}\dfrac{1}{y} = \dfrac{y}{x^2+y^2}.$

$\implies \dfrac{\partial z}{\partial y} = \dfrac{1}{1+\frac{x^2}{y^2}}\dfrac{-x}{y^2} = \dfrac{-x}{x^2+y^2}.$

$\implies \dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{-1.(x^2+y^2)-(-x)(2x)}{(x^2+y^2)^2} = \dfrac{x^2-y^2}{(x^2+y^2)^2}.$

and $\implies \frac{\partial^2 z}{\partial y\partial x} = \frac{1.(x^2+y^2)-(y)(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$

Also $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{0.(x^2+y^2)-(y)(2x)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}.$

and $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{0.(x^2+y^2)-(-x)(2y)}{(x^2+y^2)^2} = \dfrac{2xy}{(x^2+y^2)^2}.$

(Note that $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial y\partial x}$)

Example 4.17. If $u = (x^2+y^2+z^2)^{-1/2}$, then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

We have $\frac{\partial u}{\partial x} = (-1/2)(x^2+y^2+z^2)^{-3/2}2x$

$= -x(x^2+y^2+z^2)^{-3/2}.$

$\implies \frac{\partial^2 u}{\partial x^2} = (3/2)(x^2+y^2+z^2)^{-5/2}2x.x - (x^2+y^2+z^2)^{-3/2}$

$= (x^2+y^2+z^2)^{-5/2}(3x^2-(x^2+y^2+z^2))$

$= (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$

Similarly, we have

$\frac{\partial^2 u}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(2y^2-x^2-z^2)$

and $\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(2z^2-x^2-y^2)$

$$\implies \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}((2x^2-y^2-z^2)+(2y^2-x^2-z^2)+(2z^2-x^2-y^2)) = 0$$

Example 4.18. Show that if $z = xcos(y/x) + tan(y/x)$, then

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0, \quad x \neq 0$$

Given $z = xcos(y/x) + tan(y/x)$

$\implies \frac{\partial z}{\partial x} = cos(y/x) - xsin(y/x)(-y/x^2) + sec^2(y/x)(-y/x^2)$

$= cos(y/x) + (y/x)sin(y/x) - (y/x^2)sec^2(y/x) \qquad \ldots(1)$

and $\frac{\partial z}{\partial y} = -x\,sin(y/x)(1/x) + sec^2(y/x)(1/x)$

$= -sin(y/x) + sec^2(y/x)/x \qquad \ldots(2)$

Now, multiplying (1) by x and (2) by y and adding we get

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xcos(y/x) \qquad \ldots(3)$

Differentiating (3) partially w.r.t. x and y , we get

$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\partial y} = -x\,sin(y/x)(-y/x^2) + cos(y/x) \qquad \ldots(4)$

and $x\frac{\partial^2 z}{\partial y\,\partial x} + \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial y^2} = -x\,sin(y/x)(1/x)$ ...(5)
Multiplying (4) by x and (5) by y and adding, we get
$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} = xcos(y/x)$ ...(6)
From (3) and (6), we get

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

In-text Exercise 4.4. If $u = log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

4.6 Homogeneous Functions

Definition 4.2. A function $z = f(x, y)$ is said to be homogeneous function of degree n, if it can be written in the form:

$$z = x^n g\left(\frac{y}{x}\right)$$

for some function g.

For illustration we have

1. $z = ax^2 + bxy + cy^2 = x^2(a + b\frac{y}{x} + c(\frac{y}{x})^2) = x^2 g(\frac{y}{x})$ is a homogeneous function of degree 2. Here, $g(t) = a + bt + ct^2$.
2. $z = \dfrac{x^3 + y^3}{x^2 + y^2} = x\dfrac{1 + (\frac{y}{x})^3}{1 + (\frac{y}{x})^2} = xg(\dfrac{y}{x})$ is a homogeneous function of degree 1. Here $g(t) = \frac{1+t^3}{1+t^2}$

Theorem 4.3. ***(Euler's Theorem on Homogeneous Functions)***
If $z = f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

Corollary 4.1. If $z = f(x, y)$ is a homogeneous function of degree n, then it follows from the Euler's Theorem, that

(a) $x\dfrac{\partial^2 z}{\partial x^2} + y\dfrac{\partial^2 z}{\partial x\,\partial y} = (n-1)\dfrac{\partial z}{\partial x}.$

(b) $y\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial^2 z}{\partial y^2} = (n-1)\dfrac{\partial z}{\partial y}$

(c) $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\,\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = n(n-1)z$

Example 4.19. Verify Euler's Theorem for $z = tan^{-1}(y/x)$.

Solution. Clearly $z = x^0 tan^{-1}(y/x)$ is a homogeneous function of degree $n = 0$.

To verify Euler's Theorem we need to show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nz$.

Now $\dfrac{\partial z}{\partial x} = \dfrac{-y/x^2}{1+(y/x)^2} = \dfrac{-y}{x^2+y^2}$.

and $\dfrac{\partial z}{\partial y} = \dfrac{1/x}{1+(y/x)^2} = \dfrac{x}{x^2+y^2}$.

Therefore, $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = -\dfrac{xy}{x^2+y^2} + \dfrac{xy}{x^2+y^2} = 0 = 0.z$

Example 4.20. If $z = \log\dfrac{x^2+y^2}{x+y}$, then prove that:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$$

Solution. Given $z = \log\dfrac{x^2+y^2}{x+y}$

$\implies u = e^z = \dfrac{x^2+y^2}{x+y} = x\dfrac{1+(y/x)^2}{1+(y/x)}$.

This implies that u is a homogeneous function of degree 1.

Therefore by Euler's Theorem, we have

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.u = e^z \qquad \ldots(1)$

Now, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial z}.\dfrac{\partial z}{\partial x} = e^z\dfrac{\partial z}{\partial x}, \quad \ldots(2)$

$\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial z}.\dfrac{\partial z}{\partial y} = e^z\dfrac{\partial z}{\partial y} \qquad \ldots(3)$

From (1), (2) and (3), we have:

$e^z(x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}) = e^z$

$\implies x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 1.$

Example 4.21. If $z = sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = tan(z).$$

Solution. Let $u = sin(z) = \left(\frac{x^2+y^2}{x+y}\right) = x\left(\frac{1+(y/x)^2}{1+y/x}\right)$

Clearly u is a homogeneous function of degree 1.

Therefore, by the Euler's theorem, we have

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.u = sin(z) \qquad \ldots(1)$

Now $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial z}.\dfrac{\partial z}{\partial x} = cos(z)\dfrac{\partial z}{\partial x}, \qquad \ldots(2)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial y} = cos(z)\frac{\partial z}{\partial y} \qquad \ldots(3)$

From (1) and (3), we have:

$cos(z)(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) = sin(z)$

$\implies x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = tan(z).$

Example 4.22. If $z = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{sin(2z)}{4}$$

Solution. Let $u = tan(z) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = x^{1/2}(\frac{1+y/x}{1+\sqrt{y/x}})$

Therefore, u is a homogeneous function of degree 1/2.

By Euler's theorem, we have

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{u}{2} = \frac{tan(z)}{2} \qquad \ldots(1)$

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x} = sec^2(z)\frac{\partial z}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial y} = sec^2(z)\frac{\partial z}{\partial y} \qquad \ldots(2)$

From (1) and (2), we have:

$sec^2(z)(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) = tan(z)/2$

$\implies x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{tan(z)}{2sec^2(z)} = \frac{2sin(z)cos(z)}{4} = \frac{sin(2z)}{4}.$

Example 4.23. If $z = tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (1-4sin^2(z))sin(2z)$$

Solution. Let $u = tan(z) = \frac{x^3+y^3}{x-y} = x^2\frac{1+(y/x)^3}{1-(y/x)}.$

This implies that u is a homogeneous function with degree 2.

Therefore by Euler's Theorem, we have

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u = 2tan(z)$

Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial x} = sec^2(z)\frac{\partial z}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z}.\frac{\partial z}{\partial y} = sec^2(z)\frac{\partial z}{\partial y} \qquad \ldots(2)$

From (1) and (2), we have:

$sec^2(z)(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}) = 2tan(z)$

$\implies x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2tan(z)}{sec^2(z)} = sin(2z) \qquad \ldots(3)$

Differentiating (3), w.r.t. x and y, we get

$\frac{\partial z}{\partial x} + x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x\partial y} = 2cos(2z)\frac{\partial z}{\partial x} \qquad \ldots(4)$

$$x\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=2cos(2z)\frac{\partial z}{\partial y} \qquad \dots(5)$$

Multiplying (4) by x and (5) by y and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^3 z}{\partial x\,\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2cos(2z)(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y})$$

$$\implies x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^3 z}{\partial x\,\partial y^2}+2sin(2z)=2cos(2z)sin(2z)$$

$$\implies x^2\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}+2xy\frac{\partial^3 z}{\partial x\,\partial y^2}=sin(2z)(2cos(2z)-1)$$

$$=sin(2z)(1-2(1-cos(2z)))=sin(2z)(1-4sin^2(z).$$

4.7 Summary

1. If $y=\frac{1}{ax+b}$, $y_n=(-1)^n n!.a^n(ax+n)^{-(n+1)}$

2. If $y=(ax+b)^m$, $y_n=\frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$.

3. If $y=\log(ax+b)$, $y_n=\frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$.

4. If $y=sin(ax+b)$, $y_n=a^n sin(ax+b+\frac{n\pi}{2})$

5. If $y=cos(ax+b)$, $y_n=a^n cos(ax+b+\frac{n\pi}{2})$.

6. If $y=e^{ax}sin(bx+c)$, $y_n=r^n e^{ax}sin(bx+c+n\theta)$, where $r=\sqrt{a^2+b^2},\theta=tan^{-1}(b/a)$.

7. If $y=e^{ax}cos(bx+c)$, $y_n=r^n e^{ax}cos(bx+c+n\theta)$, where $r=\sqrt{a^2+b^2},\theta=tan^{-1}(b/a)$.

8. (**Leibnitz's Theorem**) If $u(x)$ and $v(x)$ are any two functions having derivative up to n^{th} order, then we have:

$$(u.v)_n=u_n.v+\binom{n}{1}u_{n-1}v_1+\binom{n}{2}u_{n-2}v_2+\cdots+\binom{n}{r}u_{n-r}v_r+\cdots+\binom{n}{n-1}u_1v_{n-1}+\binom{n}{n}uv_n$$

9. (**Euler's Theorem on Homogeneous Functions**)

   If $z=f(x,y)$ is a homogeneous function of degree n, then

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz.$$

4.8 Self Assessment Exercises

1. If $y = x(x+1)log(x+1)^3$, Find y_n.

2. If $x^5 + y^5 = 5a^2x^3$, then show that
$$y_2 + \frac{6a^2x^2(a^3+x^3)}{y^9} = 0.$$

3. Show that the n^{th} derivative of $y = sin^4(x)$, is given by
$$y_n = -2^{n-1}cos(2x + \frac{n\pi}{2}) + \frac{4^n}{8}cos(4x + \frac{n\pi}{2}).$$

4. if $y = sin(mx) + cos(mx)$, show that:
$$y_n = m^n(1 + (-1)^n sin(2mx)).$$

5. Find the n^{th} derivative of $y = sin^2(x)cos^3(x)$.

6. If $y = e^{msin^{-1}(x)}$, then show that:
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

7. If $y = cos(msin^{-1}(x))$, then show that:
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$$

8. If $y = sin^{-1}(x)$, then show that:
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$

9. If $y = sin^{-1}(x)/\sqrt{1-x^2}$, then show that:
$$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0$$

10. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, then show that:
$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

11. Show that the n^{th} derivative of $xlog(x)$ is $\dfrac{(-1)^{n-2}(n-1)!}{x^{n-1}}$.

12. If $I_n = \dfrac{d^n}{dx^n}(x^n logx)$, then prove that $I_n = nI_{n-1} + (n-1)!$ Further show that
$$I_n = n!(log(x) + 1 + 1/2 + 1/3 + \cdots + 1/n)$$

13. If $y = (1-x)^{1-\alpha}e^{-\alpha x}$, then show that:

$$(1-x)y_{n+1} - (n+\alpha x)xy_n - n\alpha y_{n-1} = 0$$

14. If $y = e^{mcos^{-1}(x)}$, then show that:

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

    Find $y_n(0)$.

15. If $y = e^{\frac{x^2}{2}}cos(x)$, then show that:

$$y_{2n+2}(0) - 4ny_{2n}(0) + (2n-1)2ny_{2n-2} = 0$$

16. If $z = log(x^2+y^2)$, then show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

17. Id $z = log(x^3+y^3-x^2y-xy^2)$, then show that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{2}{x+y}.$$

18. If $u = log(x^2+y^2+z^2)$ show that

$$x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}$$

19. If $z = sin^{-1}(x/y) + tan^{-1}(y/x)$, then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

20. If $z = log(x^2+y^2) + tan^{-1}(y/x)$, then show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

21. If $z = cos^{-1}(\frac{x+y}{\sqrt{x}+\sqrt{y}})$, then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (-1/2)cot(z).$$

22. If $z = sec^{-1}(\frac{x^2+y^2}{x+y})$ then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2cot(z).$$

23. If $z = x^2tan^{-1}(y/x) - y^2tan^{-1}(x/y)$, then show that

$$\frac{\partial^3 z}{\partial x\,\partial y^2} = \frac{x^2-y^2}{x^2+y^2}.$$

24. If $z = f(r)$, where $r^2 = x^2 + y^2$, then show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f_2(r) + \frac{f_(r)}{r}$$

25. If $z = \frac{x^2y^2}{x+y}$, then show that

$$x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^3 z}{\partial x\,\partial y^2} = 2\frac{\partial z}{\partial x}$$

4.9 Solution of In-text Exercises

In-text Exercise 4.1

Solution. Given $x = sin(t), y = sin(at) \quad \ldots(1)$

$\implies \frac{dx}{dt} = cos(t), \frac{dy}{dt} = acos(at).$

$\implies y_1 = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{acos(at)}{cos(t)}.$

Differentiating w.r.t. x, we get

$y_2 = \frac{d}{dt}(\frac{acos(at)}{cos(t)})\frac{dt}{dx} \quad \ldots(2)$

$\implies y_2 = \frac{-cos(t)p^2sin(at) + pcos(at)sin(t)}{cos^2(t)}\frac{1}{cos(t)}$

$\implies cos^2(t)y_2 = -a^2sin(at) + sin(t)\frac{acos(at)}{cos(t)}$

$\implies (1 - sin^2(t))y_2 = -a^2y + xy_2$ using (1) and (2).

$\implies (1 - x^2)y_2 - xy_1 + a^2y = 0$

In-text Exercise 4.2

Solution. We have $y = xlog(\frac{x-1}{x+1}) \quad = x(log(x-1) - log(x+1))$

Differentiating w.r.t. x, we get

$y_1 = log(x-1) - log(x+1) + x(\frac{1}{x-1} - \frac{1}{x+1})$

$\implies y_1 = log(x-1) - log(x+1) + \frac{2x}{x^2-1}.$

$\implies y_1 = log(x-1) - log(x+1) + (\frac{1}{x-1} + \frac{1}{x+!})$.

Differentiating $n-1$ times we have

$y_n = (-1)^{n-2}(n-2)!\left(\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}}\right) + (-1)^{n-1}(n-1)!\left(\frac{1}{(x-1)^n} - \frac{1}{(x+1)^n}\right)$

$\implies y_n = (-1)^n(n-2)!\left(\frac{1}{(x-1^n)}((x-1)-(n-1)) - \frac{1}{(x+1)^n}((x+1)+(n-1))\right)$

$\implies y_n = (-1)^n(n-2)!\left(\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n)}\right)$

In-text Exercise 4.3

Solution. Given $y = tan^{-1}(x) \quad \ldots(1)$

Differentiating w.r.t. x, we get

$y_1 = 1/(1+x^2) \quad \ldots(2)$

Differentiating again w.r.t. x, we get

$y_2 = \frac{-2x}{(1+x^2)^2}$

$\implies (1+x^2)y_2 = -2xy_1$ using (2)

$\implies (1+x^2)y_2 + 2xy_1 = 0 \quad \ldots(3)$

Differentiating n times by using Lebnitz's theorem, we get

$(1+x^2)y_{n+2} + \binom{n}{1}2xy_{n+1} + \binom{n}{2}2y_n + 2xy_{n+1} + 2\binom{n}{1}y_n = 0$

$\implies (1+x^2)y_{n+2} + 2nxy_{n+1} + n(n-1)y_n + 2xy_{n+1} + 2ny_n = 0$

$\implies (1+x^2)y_{n+2} + 2x(n+1)y_{n+1} + n(n+1)y_n = 0 \quad \ldots(4)$

Substituting $x = 0$ in (4)

$y_{n+2}(0) = -n(n+1)y_n(0) \quad \ldots(5)$

Substituting $x = 0$ in (1) y(0) = 0 From (2), $y_1(0) = 1$

From (2), $y_2(0) = 0$

Substituting $n = 1, 2, 3, ...$ in (5), we have

$y_3(0) = -1.2y_1(0) = -2!$

$y_4(0) = -2.3y_2(0) = 0$

$y_5(0) = -3.4.y_3(0) = 4!$

$y_6(0) = -4.5y_4(0) = 0$

$y_7(0) = -5.6.y_5(0) = -6!$ and so on

In general

$$y_n(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}n! & \text{if } n \text{ is odd} \end{cases}$$

In-text Exercise 4.4

Solution. Given $u = log(x^3 + y^3 + z^3 - 3xyz)$, then

$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$,

$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$,

and $\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$

$$\begin{aligned}\implies \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} &= \frac{3(x^2+y^2+z^2)-3(xy+yz+xz)}{x^3+y^3+z^3-3xyz}\\ &= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}\\ &= \frac{3}{x+y+z}\end{aligned}$$

4.10 Suggested Readings

1. Anton, Howard, Bivens, Irl, Davis, Stephen (2013). Calculus (10th ed.). Wiley India Pvt. Ltd. New Delhi. International Student Version. Indian Reprint 2016.

2. Prasad, Gorakh (2016). Differential Calculus (19th ed.). Pothishala Pvt. Ltd. Allahabad

3. Thomas Jr., George B., Weir, Maurice D., Hass, Joel (2014). Thomas' Calculus (13thed.). Pearson Education, Delhi. Indian Reprint 2017.

Lesson - 5

Relative Extremum

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Structure

5.1 Learning Objectives

- To learn the concept of extrema
- To distinguish between local extrema nd global extrema.
- First and second derivative test for finding relative extrema.

5.2 Introduction

Extreme values (minimum or maximum) play important role in real life. For example, maximum revenue, minimum cost, maximum area of a field, minimum-maximum velocity of a article attained in a interval etc. Geometrically, the peaks and the valleys on the graph of a function represent the point of extremum (maximum or minimum). There are two kinds of extrema namely, local extrema and global extrema. In this lesson we will discuss these concepts mathematically. We willalso discuss the methods to find the extrema of a function with some applications.

5.3 Definitions

1. **Relative (local) Extrema**:

    - A function is said to have relative maximum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. The value of $f(x_0)$ is then called the relative maximum value in the interval.
    - A function is said to have relative minimum at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval. The value of $f(x_0)$ is then called the relative minimum value in the interval.
    - If f has either a relative maximum or a relative minimum at x_0,in the interval I then f is said to have a relative extremum at x_0 and $f(x_0)$ is then called the relative extremum value in I.

    The point x_0 in the above definitions is called a critical point.

2. **Global Extremum**:

    A function is said to have global extremum at a point c in an interval $I = [a, b]$ if $f(c) \geq f(x)$ for all x in the interval I. Similarly, f is said to have global minimum at c in an interval $I = [a, b]$ if $f(c) \leq f(x)$ for all x in the interval I.

    Note: If f is differentiable function on $I = [a, b]$, then f has a global extremum at $x = c \in$ I. Then either $c = a$ or $c = b$ is a point of relative extremum

Example 5.1. Consider the function $f(x) = 2x^2, x \in (-\infty, \infty)$. Then f is continuous and differentiable for all $x \in (-\infty, \infty)$. f has the least value (global minimum) at $x = 0$, which is also a local minimum. The graph of $f(x)$ is plotted in the given figure.

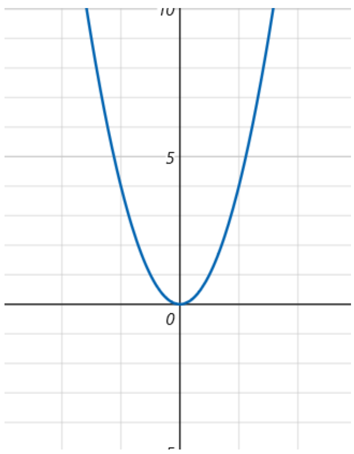

Figure 5.1: $f(x) = 2x^2$

Example 5.2. Consider the function $f(x) = -2x^2, x \in (-\infty, \infty)$. Then f is continuous and differentiable for all $x \in (-\infty, \infty)$. f has the maximum value (global maximum) at $x = 0$, which is also a local maximum. The graph of $f(x)$ is plotted in the given figure.

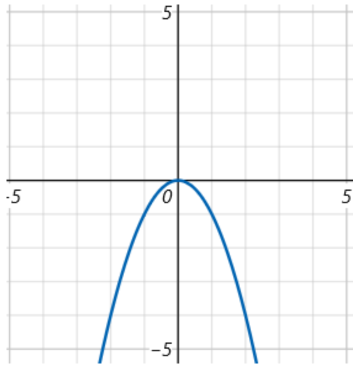

Figure 5.2: $f(x) = -2x^2$

Note: It can be noted that there are functions which do not posses relative extrema, Also it is not necessary for a function to be differentiable to possess global extrema. We illustrate these facts in the following examples.

Example 5.3. Consider the function $f(x) = x^3, x \in (-\infty, \infty)$. Then f is continuous and differentiable for all $x \in (-\infty, \infty)$ and $f'(x) = 3x^2, x \in (-\infty, \infty)$ f does not have relative extremum at $x = 0$, as there exits interval $(-1, 1)$ containing 0 such that $f(-1/2) = -1/8 < f(0) < f(1/2) = 1/8$. Therefore $f(0)$ is neither relative maximum nor relative minimum.

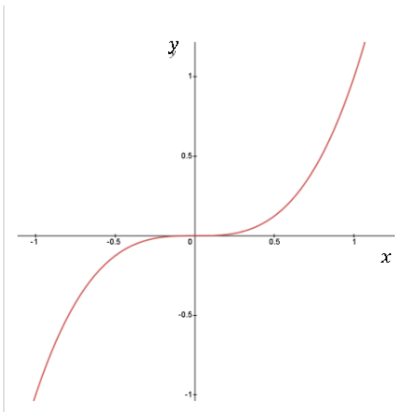

Figure 5.3: $f(x) = x^3$

Example 5.4. Consider the function $f(x) = sin(x), x \in (-\infty, \infty)$. Then f is continuous and differentiable for all $x \in (-\infty, \infty)$ and $f'(x) = cos, x \in (-\infty, \infty)$ f has the least value at $x = -\frac{3\pi}{2}, \frac{\pi}{2}$, which is a local minimum and f has the maximum value at $x = \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ which is local maximum. The graph of $f(x)$ is ploted in the given figure.

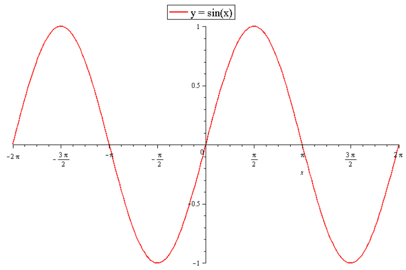

Figure 5.4: $f(x) = sin(x)$

Example 5.5. Figure 5.5 shows that $f(x) = x^3 - 3x + 5$ has relative maximum at $x = -1$ and relative minimum at $x = 1$. Therefore $x = -1, 1$ are the points of relative extrema.

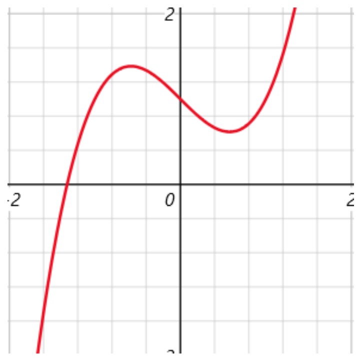

Figure 5.5: $f(x) = x^3 - 3x + 5$

Theorem 5.1. ***Necessary condition for relaive Extrema***: *Suppose that f is a function defined on an open interval containing the x_0. If f has relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f; that is either $f'(x) = 0$ or f is not differentiable at x_0.*

Example 5.6. Figure 5.6 shows that $f(x) = x^3 - 3x + 1$ has critical point at $x = 1$ and $x = -1$.
We have

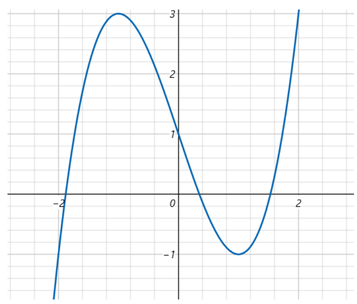

Figure 5.6: $f(x) = x^3 - 3x + 1$

$f(x) = x^3 - 3x + 1$

$\implies f'(x) = 3x^2 - 3$
$\implies 3(x^2 - 1) = 0$
$\implies x^2 - 1 = 0$
$\implies x = -1, 1$

Therefore, -1 and 1 are two critical points

Example 5.7. Figure 5.7 shows that $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$ has critical points at $x = 0$ and $x = 2$.

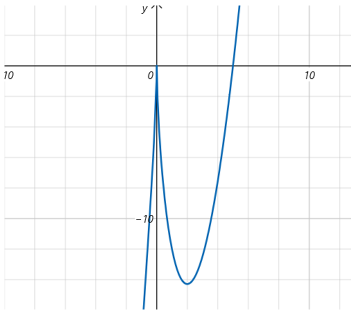

Figure 5.7: $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$

Here, we note that $f(x)$ is not differentiable at $x = 0$ and $f'(x) = 3.\frac{5}{3}x^{\frac{2}{3}} - 15.\frac{2}{3}x^{\frac{-1}{3}}$, for $x \neq 0$
Therefore $f'(x) = 0 \implies 5x^{\frac{2}{3}} - 10x^{\frac{-1}{3}} = 0$
$\implies x = 2$

Hence $x = 0$ and $x = 2$ are the critical points

Example 5.8. Figure 5.8 shows that $f(x) = 2|x|$ has critical point at $x = 0$.

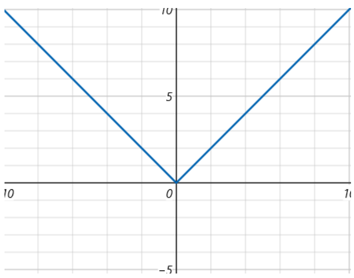

Figure 5.8: $f(x) = 2|x|$

Here, we note that $f(x) = 2|x|$ is not differentiable at $x = 0$, therefore $x = 0$ is the critical point whereas f is differentiable at every real number other then 0. Also $f(x) = |x|$ has only one critical point, namely 0.

We now present the same test to determine the local(relative) extrema of differential functions.

5.4 First Derivative Test

Theorem 5.2. *Let $y = f(x)$ be a differential function. Then $f(x)$ has a relative extremum at $x = c$ if and only if $f'(x)$ changes sign as x passes through c.*

a) If $f'(x)$ changes sign from positive to negative as x passes through c from left to right, then $f(x)$ has relative maximum at $x = c$.

b) If $f'(x)$ changes sign from negative to positive as x passes through c from left to right, then $f(x)$ has relative minimum at $x = c$.

c) If $f'(x) = 0$ does not change sign as x passes through c, then $f(x)$ does not have a relative extremum at c.

Remark.
1. If f is increasing on the left side of x_0 and decreasing on right side of x_0, then f has relative maximum at x_0.
2. If f is decreasing on the left side of x_0 and increasing on right side of x_0, then f has relative minimum at x_0.
3. If f has same behaviour on both side of x_0 that is, f is increasing on both side or decreasing on both side, then f does not have a relative extremum at x_0

Example 5.9. Figure 5.9 shows that $f(x) = 10x^2$ is increasing on right side of $x = 0$ and decreasing on left side from $x = 0$, then $x = 0$ is a point of relative minimum or f has relative minimum at $x = 0$.
We have $f(x) = 10x^2$

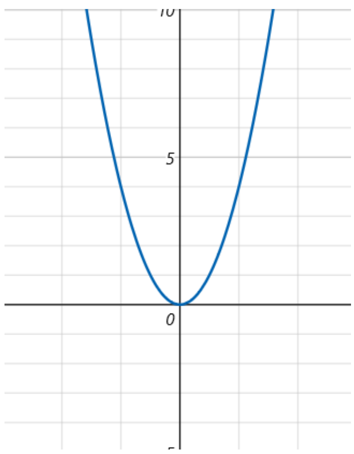

Figure 5.9: $f(x) = 10x^2$

$\implies f'(x) = 20x$
$\implies f'(x) > 0$ on $(0, \infty)$
and $f'(x) < 0$ on $(-\infty, 0)$

Example 5.10. Figure 5.10 shows that $f(x) = -10x^2$ is increasing on left side of $x = 0$ and decreasing on right side from $x = 0$, then $x = 0$ is a point of relative maximum or f has relative minimum at $x = 0$.
we have, $f(x) = -10x^2$

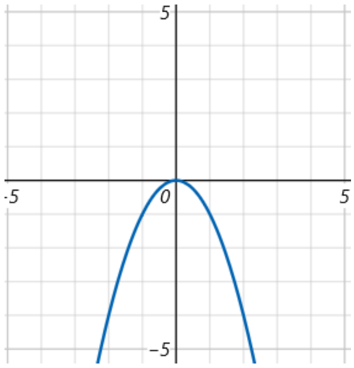

Figure 5.10: $-10x^2$

$f'(x) = -20x$
Therfore, $f'(x) < 0$ on $(0, \infty)$
and $f'(x) > 0$ on $(-\infty, 0)$

Example 5.11. Figure 5.11 shows that $f = x^3$ is increasing on both side of $x = 0$, then $x = 0$ is not a point of relative extremum or f does not have relative extremum at $x = 0$.
here, $f'(x) = 3x^2$, $x \in (-\infty, \infty)$

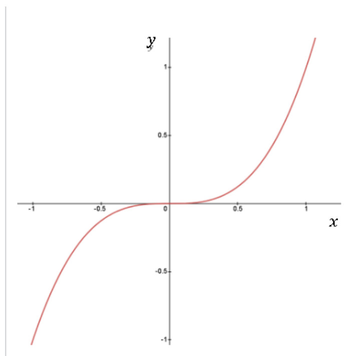

Figure 5.11: $f = x^3$

That is $f'(x)$ has the same sign in every open interval containing 0.
or $f'(x) > 0$ on $(-\infty, \infty)$

Example 5.12. Figure 5.12 shows that $f(x) = -x^3$ is decreasing on both side of $x = 0$, then $x = 0$ is not a point of relative extremum or f does not have relative extremum at $x = 0$.
Here, $f'(x) = -3x^2$
$\implies f'(x) < 0$ on $(-\infty, \infty)$

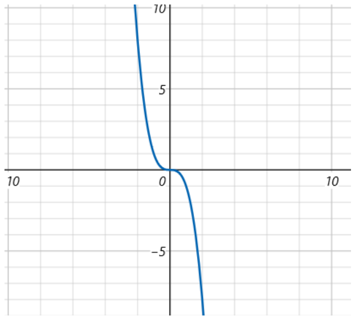

Figure 5.12: $f(x) = -x^3$

Example 5.13. Figure 5.13 shows that $f(x) = x^2 - 2x + 10$ is increasing on right side of $x = 1$ and decreasing on left side from $x = 1$. Therefore, $x = 1$ is a point of relative minimum or f has relative minimum at $x = 1$.

Solution. We have, $f(x) = x^2 - 2x + 10$
$\implies f'(x) = 2x - 2$

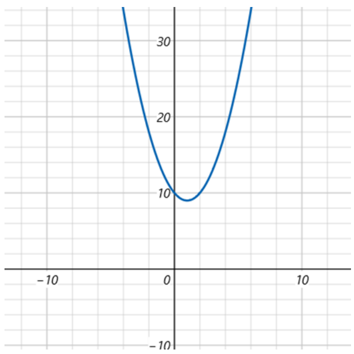

Figure 5.13: $f(x) = x^2 - 2x + 10$

Now, $f'(x) = 0$
$\implies 2x - 2 = 0$
$\implies 2x = 2$
$\implies x = 1$
Therefore, $f'(x) > 0$ on $(0, \infty)$
and $f'(x) < 0$ on $(-\infty, 1)$

Example 5.14. Figure 5.14 shows that $f(x) = x^3 + 3x^2 + 10$ is increasing on left side of $x = -2$, decreasing on right side of $x = -2$. Also $f(x)$ is decreasing on left hand side of $x = 0$ and increasing on right hand side of $x = 0$.Hence, at $x = -2$, $f(x)$ has relative maximum and at $x = 0$, $f(x)$ has relative minimum.

Solution. We have, $f(x) = x^3 + 3x^2 + 10$
$f'(x) = 3x^2 + 6x$
Therefore, $f'(x) = 0$
$\implies 3x^2 + 6x = 0$

$\implies 3x(x+2) = 0$
$\implies x = 0, -2$
Now $f'(x) > 0 \quad on \quad (-\infty, -2)$
, $f'(x) < 0 \quad on \quad (-2, 0)$
, and $f'(x) > 0 \quad on \quad (0, \infty)$

Figure 5.14: $f(x) = x^3 + 3x^2 + 10$

In-text Exercises 5.1

Find the relative extremum using first derivative test.

1. $f(x) = 1 + 8x - 3x^2$
2. $f(x) = \sin 2x \qquad 0 < x < \pi$
3. $f(x) = x^4 - 4x^3 + 4x^2$

5.5 Second Derivative Test

Theorem 5.3. *Suppose that f is twice differentiable at the point x_0 and $f'(x_0) = 0$. Then*

a) f has relative minimum at x_0 if $f''(x) > 0$.

b) f has relative maximum at x_0 if $f''(x) < 0$.

c) The test is inconclusive, that is, f may or may not have a relative maximum, a relative minimum, at x_0 if $f''(x) = 0$

Example 5.15. Find the relative extrema of $f(x) = 3x^5 - 5x^3$, $x \in R$

Solution. We have, $f(x) = 3x^5 - 5x^3$
$\implies f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$
and $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$
Now, $f'(x) = 0 \implies 15x^2(x^2 - 1) = 0$
$\implies x = 0, -1, 1$ are critical points
Now, $f''(0) = 60(0)^3 - 30(0) = 0$
Therefore, the test is inconclusive to examine the extremum at $x = 0$. In this case, we go to first derivative test.
$f''(-1) = 60(-1)^3 - 30(-1) = -30 < 0$
$\implies f$ has relative maximum at x=-1
$f''(1) = 60(1)^3 - 30(1) = 30$
$\implies f$ has relative minimum at x=1

Example 5.16. Find the relative extremum of $f(x) = x^3 - 3x + 2$, $x \in R$

Solution. We have, $f(x) = x^3 - 3x + 2$
$\implies f'(x) = 3x^2 - 3 = 3(x^2 - 1)$
and $f''(x) = 6x$
Now, $f'(x) = 0 \implies 3(x^2 - 1) = 0$
$\implies x^2 - 1 = 0$
$\implies x = \pm 1$
$f''(-1) = 6(-1) = -6 < 0$
$\implies f$ has relative maximum at $x = -1$ and relative maximum is $f(-1) = 4$
and $f''(1) = 6(1) = 6 > 0$
$\implies f$ has relative minimum at $x = 1$ and the relative minimum is $f(1) = 0$

Example 5.17. Find the relative extremum of $f(x) = x^2$

Solution. $f(x) = x^2$
$f'(x) = 2x$
$f''(x) = 2$
$f'(x) = 0 \implies 2x = 0 \implies x = 0$
$f''(x) = 2 > 0$
$\implies f$ has relative minimum at $x = 0$.

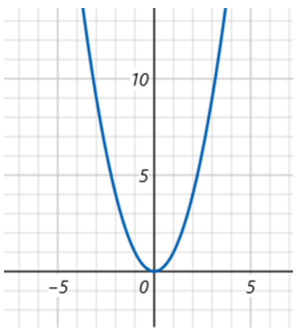

Figure 5.15: $f(x) = x^2$

Example 5.18. Find the relative extremum of $f(x) = -x^2$

Solution. $f(x) = -x^2$
$f'(x) = -2x$
$f''(x) = -2$
$f'(x) = 0 \implies -2x = 0 \implies x = 0$
$f''(x) = -2 < 0$
$\implies f$ has relative maximum at $x = 0$.

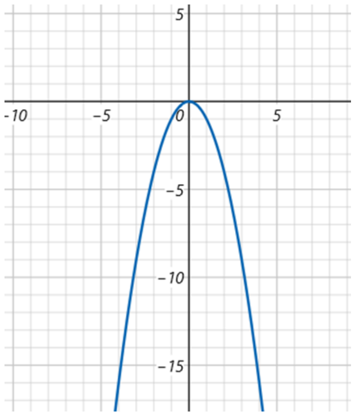

Figure 5.16: $f(x) = -x^2$

Example 5.19. Find the relative extremum of $f(x) = x^3$

Solution. $f(x) = x^3$
$f'(x) = 3x^2$
$f''(x) = 6x$
$f'(x) = 0 \implies 3x^2 = 0 \implies x = 0$
$f''(x) = 6.0 = 0$
$\implies f$ does not have relative extremum at $x = 0$.

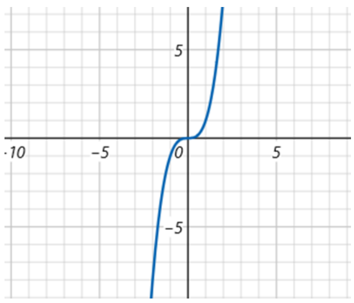

Figure 5.17: $f(x) = x^3$

Example 5.20. Find the relative extremum of $f(x) = -x^3$

Solution. $f(x) = -x^3$
$f'(x) = -3x^2$
$f''(x) = -6x$
$f'(x) = 0 \implies -3x^2 = 0 \implies x = 0$
$f''(x) = -6.0 = 0$
$\implies f$ does not have relative extremum at $x = 0$.

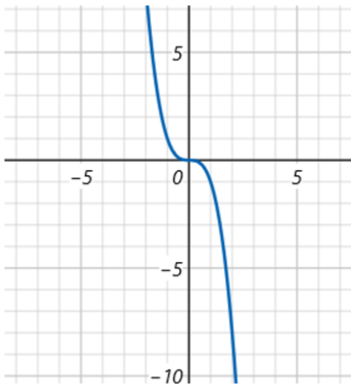

Figure 5.18: $f(x) = -x^3$

Example 5.21. Find the relative extremum of $f(x) = x^2 - 2x + 10$

Solution. We have, $f(x) = x^2 - 2x + 10$
$\implies f'(x) = 2x - 2$
and $f''(x) = 2$
Now, $f'(x) = 0 \implies 2x - 2 = 0 \implies x = 1$
$f''(x) = 2 > 0$
$\implies f$ has relative minimum at $x = 1$ and the relative minimum is 9.

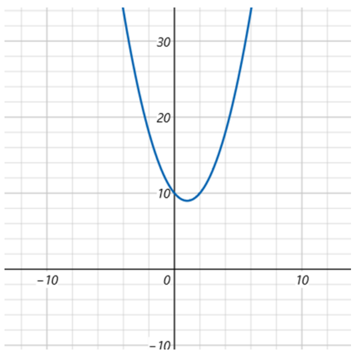

Figure 5.19: $f(x) = x^2 - 2x + 10$

Example 5.22. Find the relative extremum of $f(x) = x^3 + 3x^2 + 10$

Solution. $f(x) = x^3 + 3x^2 + 10$
$f'(x) = 3x^2 + 6x$
$f''(x) = 6x + 6$
$f'(x) = 0 \implies 3x^2 + 6x = 0$
$\implies 3x(x + 2) = 0$
$\implies x = 0, -2$ $f''(x) = 6.0 + 6 > 0$
$\implies f$ has relative minimum at $x = 0$
. $f''(x) = 6.(-2) + 6 = -6 < 0$
$\implies f$ has relative maximum at $x = -2$

Figure 5.20: $f(x) = x^3 + 3x^2 + 10$

In-text Exercise 5.2

Find the relative extremum using second derivative test

1. $f(x) = 2x^3 - 9x^2$ on $I = [-1, 5]$
2. $f(x) = (x + 1)^{\frac{2}{3}} + 2$
3. $f(x) = 2x^3 + 3x^2 - 12x$
4. $f(x) = \sqrt{1 - x^2}$

5.6 Summary

1. A function is said to have relative maximum at x_0 if $f(x_0) \geq f(x)$ for all $x \in I$, where I is an open interval.
2. A function is said to have relative minimum at x_0 if $f(x_0) \leq f(x)$ for all$x \in I$, where I is an open interval.

3. A function is said to have relative extremum if it has either relative maximum or relative minimum.

4. $x = x_0$ is a critical point of f, if either $f'(x_0) = 0$ or f is not differentiable at x_0.

5. f is continuous at critical point x_0.

   i $f'(x) > 0$ on left side of x_0 and $f'(x) < 0$ on right side of x_0, then f has relative maximum at x_0.

   ii $f'(x) < 0$ on left side of x_0 and $f'(x) > 0$ on right side of x_0, then f has relative minimum at x_0.

   iii $f'(x)$ has same sign on both side, then f does not have relative extremum at x_0.

6. Suppose f is twice differentiable at x_0.

   i If $f'(x_0) = 0 \ \ and \ \ f''(x_0) > 0$, then f has relative minimum at x_0).

   ii If $f'(x_0) = 0 \ \ and \ \ f''(x_0) < 0$, then f has relative maximum at x_0).

   iii If $f'(x_0) = 0 \ \ and \ \ f''(x_0) = 0$, then test is inconclusive.

5.7 Self Assessment Exercises

Find the relative extremum using both first and second derivative test.

1. $f(x) = 1 + 8x - 3x^2$

2. $f(x) = \sin 2x \qquad 0 < x < \pi$

   Find relative extremum using any methods.

3. $f(x) = x^4 - 4x^3 + 4x^2$

4. $f(x) = x^3(x + 1)^2$

5. $f(x) = 2x + 3x^{\frac{2}{3}}$

6. $f(x) = \frac{x+3}{x-2}$

7. $f(x) = |3x - x^2|$

5.8 Solution to In-Text Exercises

Exercise 5.1

1. Relative Maximum at $(\frac{4}{3}, \frac{19}{3})$

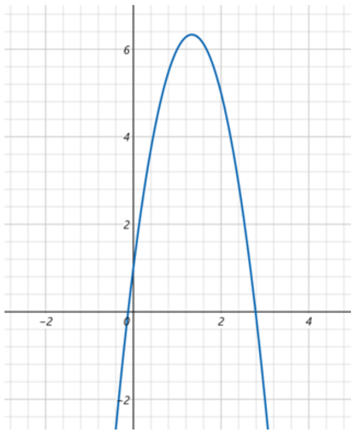

Figure 5.21: $f(x) = 1 + 8x - 3x^2$

2. Relative Maximum at $(\frac{\pi}{4}, 1)$
   Relative Minimum at $(\frac{3\pi}{4}, -1)$

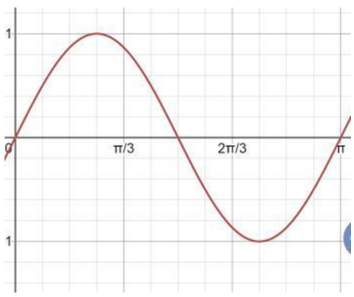

Figure 5.22: $f(x) = \sin 2x \qquad 0 < x < \pi$$

3. Relative Maximum at $(1, 1)$
   Relative Minimum at $(0, 0)$, $(2, 0)$

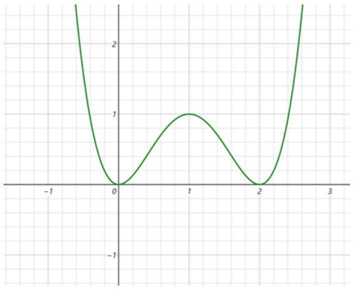

Figure 5.23: $f(x) = x^4 - 4x^3 + 4x^2$

Exercise 5.2

1. Relative Maximum at $(0, 0)$
   Relative Minimum at $(3, -27)$

2. Relative Minimum at $(1, 2)$

3. Relative Minimum at $(1, -7)$

4. Relative Maximum at $(0, 1)$
   Relative Minimum at $(-1, 0)(1, 0)$

Solution of Self Assessment Exercises

1. Relative Maximum at $(\frac{4}{3}, \frac{19}{3})$

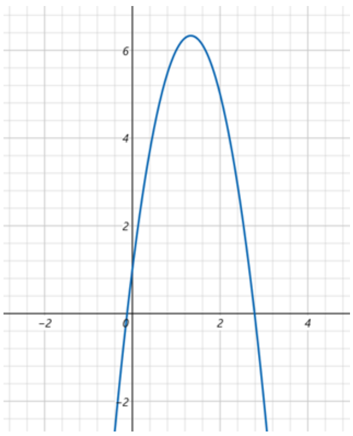

Figure 5.24: $f(x) = 1 + 8x - 3x^2$

2. Relative Maximum at $(\frac{\pi}{4}, 1)$
   Relative Minimum at $(\frac{3\pi}{4}, -1)$

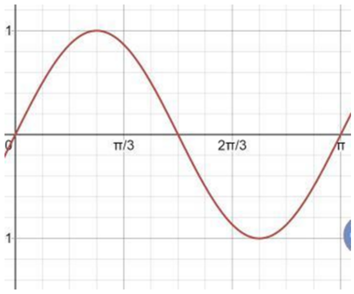

Figure 5.25: $f(x) = \sin 2x \qquad 0 < x < \pi\$$

3. Relative Maximum at $(1, 1)$
   Relative Minimum at $(0, 0), (2, 0)$

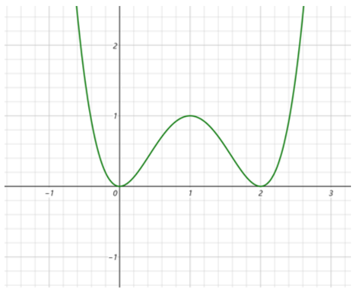

Figure 5.26: $f(x) = x^4 - 4x^3 + 4x^2$

4. Relative Maximum at $(-1, 0)$
   Relative Minimum at $(-\frac{3}{5}, -\frac{108}{125})$

Figure 5.27: $f(x) = x^3(x + 1)^2$

5. Relative Maximum at $(-1, 1)$
   Relative Minimum at $(0, 0)$

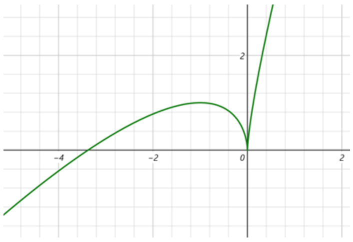

Figure 5.28: $f(x) = 2x + 3x^{\frac{2}{3}}$

6. No Relative Extremum

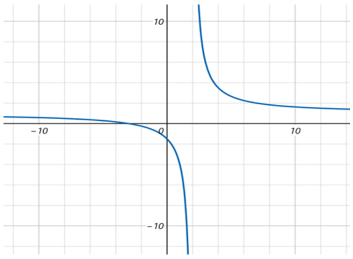

Figure 5.29: $f(x) = \frac{x+3}{x-2}$

7. Relative Maximum at $(\frac{3}{2}, \frac{9}{4})$
   Relative Minimum at $(0, 0)$, $(3, 0)$

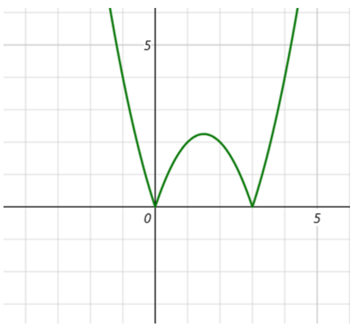

Figure 5.30: $f(x) = |3x - x^2|$

Lesson - 6

Mean Value Theorems and their Applications

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Structure

6.1 Learning Objectives

- To understand and learn the Rolle's Theorem and its applications.
- To learn the geometrical interpretation of Rolle's Theorem.
- To understand the Lagrange's Mean Value Theorem.
- To understand the geometrical interpretation of Lagrange's Mean Value Theorem and its applications.
- To understand the concept of monotonicity of functions.
- To prove some important inequalities.

6.2 Introduction

In the previous lessons, we have learnt about the limit, continuity and derivative of a function and also learnt some applications of these concepts in analysis, calculus and other areas of mathematics. In this lesson, we shall study about the mean value theorems, which is another application of derivative. Here, we shall show that the mean value theorems lead to the theorems, which are extensively used to study geometrically as well as analytically the properties of functions such as monotonicity, maxima-minima etc. Interestingly, we will notice that mean value theorems assert the existence of a particular point in a given domain (interval) having some desirable property, but these theorems do not tell us how to find that point.

In this lesson, we shall study and discuss Rolle's theorem and Lagrange's Mean Value Theorem, both playing vital role in the field of calculus. We will also learn how Lagrange's Mean Value Theorem leads to some techniques which deal with the monotonicity of functions and inequalities.

6.3 Mean Value Theorems

In this section, we will discuss Rolle's Theorem and Lagrange's Mean Value Theorem. We start this section with an example which will provide us a rough idea about the Mean Value Theorems.

Suppose a car is moving with an average speed of 50 km/h, then we can say that instantaneous speed of the car cannot be more than 50 km/h always. Similarly, it cannot be always less than 50. But one can observe that, there always exists at least at one moment, when its instantaneous speed is exactly 50 km/h. This phenomenon, where the mean value relates with the actual values is known as **Mean Value Theorem**.

In the remaining lesson, we shall study that Rolle's Theorem and Lagrange's Mean Value Theorem provide some tools to prove other results which are useful in many branches of mathematics, like calculus, analysis, optimization etc.

6.4 Rolle's Theorem

Rolle's Theorem is named after a French mathematician Michel Rolle, but interestingly, the present form of Rolle's Theorem was not actually proved by him. He had only stated and proved the theorem for the polynomial functions only. It was Cauchy, another French mathematician, who first provided the proof of Rolle's Theorem as a corollary of the Mean Value Theorem.

Now, we shall provide the formal statement of Rolle's Theorem as follows;

Theorem 6.1 (**Rolle's Theorem**). *Let f be a real-valued function defined on a closed and bounded interval $[a,\ b]$ satisfying the following properties:*

1. *f is continuous on $[a, b]$;*
2. *f is differentiable on $(a,\ b)$;*

3. $f(a) = f(b)$

Then there exists at least a point $c \in (a, b)$ *such that*

$$f'(c) = 0.$$

Proof. Let f be a real-valued function which is continuous on a given closed and bounded interval $[a, b]$ and differentiable on $(a,\ b)$ such that $f(a) = f(b)$. Then, we can consider the following cases:

1. If f is a constant function, that is $f(x) = f(a)$ for all $x \in [a,\ b]$. Then, we have

$$f'(x) = 0 \text{ for all } x \in [a, b]$$

   Hence, we get a $c \in (a, b)$ such that $f'(c) = 0$.

2. Suppose f is not a constant function. As f is given to be continuous on $[a, b]$, therefore either f will increase or decrease as $x > a$. Without loss of generality, let us suppose that f is increasing for $x > a$. Since, $f(a) = f(b)$, therefore f will increase till some values of $x = c > a$ (say) and then start to decrease for $x > c$. As f is differentiable for all $x \in (a,\ b)$, therefore f must attain its extrema (local maxima) at $x = c$.

Obviously, we have $f(a) \neq f(c)$ and $a \neq c$ and $c \neq b$, that is, $c \in (a, b)$. As f attains its maxima at $x = c$, thus for small positive number $h > 0$, we have

$$f(c+h) - f(c) \leq 0 \text{ and } f(c-h) - f(c) \leq 0$$

Therefore,

$$\frac{f(c+h) - f(c)}{h} \leq 0 \text{ and } \frac{f(c-h) - f(c)}{-h} \geq 0 \tag{6.1}$$

$$Rf'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0 \text{ and } Lf'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \geq 0.$$

Since f is differentiable for all $x \in (a, b)$, therefore $Rf'(c) = Lf'(c)$ and hence $f'(c) = 0$. Hence the proof. □

Geometrical Interpretation of Rolle's Theorem

Geometrically, Rolle's Theorem claims that if a real-valued function f is defined over a closed and bounded interval $[a, b]$ which is continuous on $[a, b]$ and differentiable on (a, b), that is, the function f can be drawn smoothly without any gaps, jumps and sharp edges. Also, the end points of the curve lie on a horizontal line, that is $f(a) = f(b)$. Then somewhere between a and b, the curve $y = f(x)$ has a horizontal tangent, that is, there exists a point $c \in (a, b)$ such that the tangent at $(c, f(c))$ is parallel to x-axis. Also, the proof of the theorem suggests that this happens at the point where the function attains its extrema.

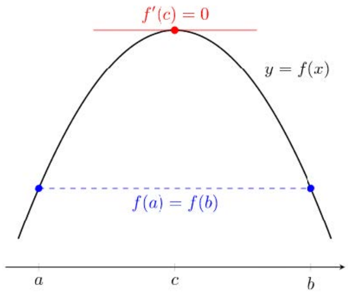

Figure 6.1: Geometrical Interpretation of Rolle's Theorem

Example 6.1. Verify the Rolle's Theorem for

$$f(x) = \cos^2 x \quad \text{on } [0,\ 2\pi].$$

Solution. Consider the function $f(x) = \cos^2 x$ defined on $[0,\ 2\pi]$. Since the functions $g(x) = \cos x$ and $h(x) = x^2$, both are continuous functions on $[0,\ 2\pi]$, being trigonometric and polynomial functions and composition of two continuous functions is again continuous. Thus $f(x) = h \circ g(x)$ is continuous on $[0,\ 2\pi]$.
Also,

$$f'(x) = -2\cos x \sin x = -\sin 2x$$

exists for all $x \in [0, 2\pi]$. Therefore $f(x)$ is differentiable on $(0,\ 2\pi)$.
Also,

$$f(0) = \cos^2(0) = 1 \text{ and } f(2\pi) = \cos^2(2\pi) = 1$$

Hence, $f(0) = 1 = f(2\pi)$.
Since all the conditions of Rolle's Theorem hold true, therefore there exists a real number $c \in (0,\ 2\pi)$ such that $f'(c) = 0$, that is $\sin 2c = 0$.
Hence, $c = \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}, \ldots$

One can observe the same from the figure 6.2 below.
One can also observe that at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, the given function attains it minimum value and at $x = \pi$, it attains the maximum value.

Remark. The Rolle's Theorem asserts the existence of at least one point $c \in (a, b)$ such that $f'(c) = 0$ if the hypothesis holds true. Infact, more than one such points may exist, as in Example 6.1.

Corollary 6.1. Let f be a real-valued function, which is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$. Let a and b be two zeroes of f. Then, by the Rolle's Theorem, there exists at least one $c \in (a,\ b)$ such that $f'(c) = 0$.

That is, geometrically, we can say that between every pair of zeroes of f, there always exists a zero of f', provided that f is continuous and differentiable.
Now, we shall show that all the three conditions given in the Rolle's Theorem are necessary. One cannot drop any of them and still get the desired result.
In the next example, we will illustrate that the assumption of continuity of f in Rolle's Theorem cannot be relaxed.

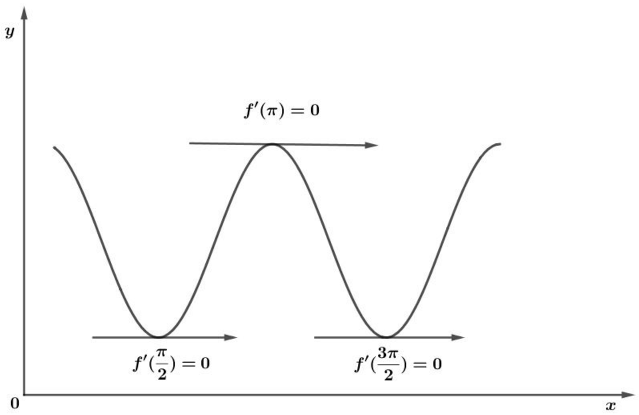

Figure 6.2: $f(x) = \cos^2 x$

Example 6.2. Let us consider a function $f\,(x) = \tan x$ on $[0,\ \pi]$. Examine whether Rolle's Theorem holds.

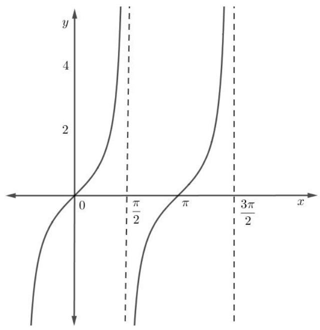

Figure 6.3: $y = \tan x$

Solution. Let $f\,(x) = \tan x$, $x \in [0,\ \pi]$. Then, we have

$$f\,(0) = \tan{(0)}\ = 0 \text{ and } f\,(\pi) = \tan{(\pi)}\ = 0.$$

Also, $f'\,(x) = \sec^2 x.$

$$\text{Now, } f'\,(c) = 0\ \Rightarrow \sec^2 c = 0$$

Thus, we have $\sec c\ = 0.$

Since the range of the function $\sec x$ is $(-\infty, -1] \bigcup [1, \infty)$. Therefore, there does not exist any $c \notin (0,\ \pi)$ such that $f'(c) = 0.$
Hence, the conclusion of Rolle's Theorem does not hold good.

Here, we can observe that $f(x) = \tan x$ is not continuous at $x = \frac{\pi}{2}$. Thus, Rolle's Theorem does not hold true.
In the next example, we will see that the condition of differentiability in the hypothesis of Rolle's Theorem is necessary.

Example 6.3. Examine the applicability of Rolle's Theorem to the function

$$f(x) = |x| \quad \text{for } x \in [-1, 1].$$

Solution. Consider the function $f(x) = |x|$, $x \in [-1, 1]$. Then, $f(-1) = 1 = f(1)$. Also, f is continuous on $[-1, 1]$. But for $x = 0$, f is not differentiable. Hence, f is not differentiable on $(-1, 1)$.
Thus, only two out of three conditions of Rolle's Theorem hold good.

$$f'(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}.$$

Therefore, there does not exist any $c \in (-1,\ 1) \setminus \{0\}$ such that $f'(c) = 0$. Hence Rolle's Theorem is neither applicable nor its conclusion hold.

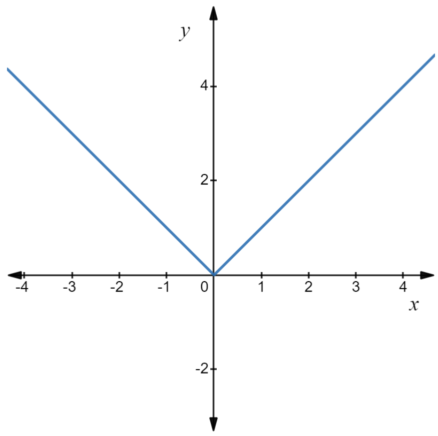

Figure 6.4: $y = |x|$

In the next example, we will demonstrate that the condition of $f(a) = f(b)$ is essential for the Rolle's Theorem to hold.

Example 6.4. Verify whether the Rolle's Theorem is applicable to

$$f(x) = x \quad \text{for } x \in [-1, 1].$$

Solution. Here, the function $f(x) = x$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.
Also

$$f'(x) = 1 \text{for all } x \in (-1, 1).$$

$\Rightarrow f'(x) \neq 0$ for any $x \in (-1, 1)$.

Hence, the Rolle's Theorem is not applicable for $f(x) = x$ on $[-1, 1]$.

In the following examples, we illustrate the Rolle's Theorem and its applications.

Example 6.5. Verify Rolle's Theorem for the function

$$f(x) = \sin x + \cos x \quad \text{on} \quad \left[0, \frac{\pi}{2}\right].$$

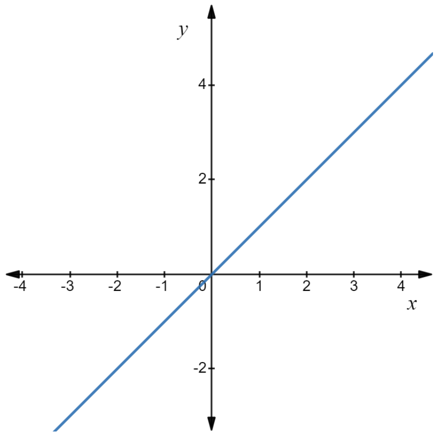

Figure 6.5: $y = x$

Solution. Consider the function $f\,(x) = \sin x\, + \cos x$ on $\left[0, \frac{\pi}{2}\right]$. Since the cosine and sine functions are continuous and differentiable, therefore f is also continuous on $\left[0, \frac{\pi}{2}\right]$ and differentiable on $\left(0, \frac{\pi}{2}\right)$.
Also, we have

$$f\,(0) = \sin\,(0)\, + \cos\,(0)\, = 1$$

and

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = 1.$$

Thus, $f\,(0) = f\left(\frac{\pi}{2}\right) = 1$.
Since, all the conditions of Rolle's Theorem are hold true, thus there must exists some $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'\,(c) = 0$.

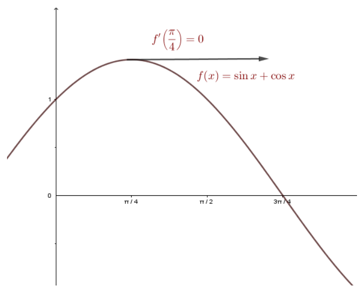

Figure 6.6: $f(x) = \sin x + \cos x$

Now, $f'(x) = \cos x - \sin x$. Therefore, $f'(x) = 0$ if $\cos x - \sin x = 0$. Thus, we have, $\sin x = \cos x$ or $\tan x = 1$. Therefore, there exists $c = \frac{\pi}{4} \in (0, \frac{\pi}{2})$ such that

$$f'(c) = \tan(\frac{\pi}{4}) = 1$$

Hence Rolle's Theorem is verified.

Example 6.6. Verify the Rolle's Theorem for $f\,(x) = (x^2 + 2x - 3)\,e^x$ on $[-3,\ 1]$.

Solution. Consider the function

$$f(x) = \left(x^2 + 2x - 3\right) e^x \quad \text{on} \ [-3,\ 1]$$

As polynomial and exponential functions, both are continuous and differentiable, therefore $f(x)$, being the product function of the two functions. It is also continuous on $[-3,\ 1]$ and differentiable on $(-3,\ 1)$. Also, $f(-3) = \left((-3)^2 + 2(-3) - 3\right) e^{-3} = 0$ and $f(1) = \left((1)^2 + 2(1) - 3\right) e^{-1} = 0$.
Since, all the conditions of Rolle's Theorem hold, there must exist some $c \in (-3,\ 1)$ such that

$$f'(c) = 0.$$

Now

$$\begin{aligned} f'(x) &= (2x+2)\, e^x + \left(x^2 + 2x - 3\right) e^x \\ &= (x^2 + 4x - 1)e^x \\ &= 0 \end{aligned}$$

Thus, we have $\Rightarrow x^2 + 4x - 1 = 0$ as $e^x \neq 0$
$\Rightarrow x = -2 \pm \sqrt{5}$

Hence, there exists $c = 2 + \sqrt{5} \in (-3, 1)$ such that $f'(c) = 0$. Thus, the Rolle's Theorem is verified.

Theorem 6.2 (Intermediate Value Theorem). *Suppose that f is a continuous function on a closed interval $[a, b]$ with $f(a) \neq f(b)$. If M is a real number between $f(a)$ and $f(b)$, then there is a number c in the interval (a, b) such that*

$$f(c) = M.$$

One of the important application of Intermediate Value theorem is the existence of the root of a given equation.

Definition 6.1. Let $f(x)$ be a real valued function defined on $\mathbb{R}$. then a point $a \in \mathbb{R}$ is called a *root* of f if $f(a) = 0$.

Theorem 6.3. ***(Location of roots)*** *Let f be a continuous function defined on the interval $[a, b]$ such that $f(a)$ and $f(b)$ are of different sign, that is, $f(a).f(b) < 0$ then, there is a root $c \in (a, b)$ of f.*

The Theorem 6.2 and 6.3, have also been discussed in Chapter 2.

Example 6.7. Show that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.

Solution. Consider the function $f(x) = x^{13} + 7x^3 - 5$. Here, $f(0) = -5 < 0$ and $f(1) = 1 + 7 - 5 = 3 > 0$. Since, being a polynomial, f is continuous and also, $f(0).f(1) < 0$, thus, by Theorem 6.3, there must exist some $c \in (0, 1)$ such that $f(c) = 0$.
Also, one can observe that $f(x) < 0$ for all $x < 0$. Now, let if possible, there exists two distinct positive roots say a and b of the given equation. Therefore, we have $f(a) = f(b) = 0$. Thus, by the Rolle's Theorem, there exists some $c \in (a, b)$ such that

$$f'(c) = 0.$$

that is,

$$13c^{12} + 21c^2 = 0$$

$$c^2\left(13c^{10} + 21\right) = 0$$

We get $c = 0$, which leads to contradiction as $c \in (a, b)$ and $a > 0$. Hence the result.

Example 6.8. Show that there is no real number k such that the equation $x^3 - 3x + k = 0$ has two distinct roots in $[0, 1]$.

Solution. Let, if possible, there exists some value of k for which the given equation $x^3 - 3x + k = 0$ has two distinct roots say a and b in $[0, 1]$.
Without loss of generality, we assume that $b > a$.
Now, consider $f(x) = x^3 - 3x + k$ on $[a,\ b]$. Since the function f is continuous on $[a,\ b]$ and differentiable on (a, b), being a polynomial. Also, $f(a) = f(b) = 0$. Therefore, by the Rolle's Theorem, there exists some $c \in (a,\ b) \subseteq [0,\ 1]$ such that

$$f'(c) = 0$$

that is, $3c^2 - 3c = 0$. Hence, we get $c = \pm 1$, which is a contradiction as $c \in (a, b) \subseteq [0, 1]$. Hence our assumption is wrong. Thus, the desired result.

Example 6.9. Verify whether Rolle's Theorem is applicable to the function

$$f(x) = x(x+3)e^{-\frac{1}{2}x} \quad \text{on} \quad [-3,\ 0].$$

Solution. Consider

$$f(x) = x(x+3)e^{-\frac{1}{2}x}$$

We have, $f(-3) = f(0) = 0$. Also,

$$f'(x) = (2x+3)e^{-\frac{1}{2}x} + x(x+3)e^{-\frac{1}{2}x}\left(-\frac{1}{2}\right)$$

$$= \frac{(-x^2 + x + 6)}{2}e^{-\frac{1}{2}x}$$

For $x \in (-3,\ 0)$, $f'(x)$ is well defined and hence f is differentiable on $(-3,\ 0)$ and continuous on $[-3,\ 0]$.
Now

$$f'(x) = 0 \Rightarrow -x^2 + x + 6 = 0.$$

Thus, we have $x = -2, 3$. Therefore, there exists $c = -2 \in (-3, 0)$ such that $f'(c) = 0$. Hence, Rolle's Theorem is verified.

In-text Exercise 6.1. 1. *Verify Rolle's Theorem in the interval* $[a,\ b]$ *for the functions:*

(a) $(x-a)^m(x-b)^n$; $m,\ n$ *being positive integers;*
(b) $\log\frac{x^2+ab}{(a+b)x}$.

2. *Verify Rolle's Theorem for the functions:*

(a) $\frac{\sin x}{e^x}$ *on* $[0,\ \pi]$;

(b) $e^x(\sin x\ - \cos x\ )$ *in* $\left[\frac{\pi}{4}, \frac{5\pi}{4}\ \right]$;

3. *In each of the following, determine whether Rolle's Theorem is applicable. If applicable, find the appropriate point in the interval where the derivatives vanish. If not applicable, explain the reason.*

   (a) $f\left(x\right) = |x-2|$ *on* $[0,\ 4]$;

   (b) $g\left(x\right) = \cos x$ *on* $[0,\ 10\pi]$;

   (c) $h\left(x\right) = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ *on* $[0,\ 1]$;

   (d) $j\left(x\right) = x^3 + x$ *on* $[0,\ 1]$

   (e) $k\left(x\right) = x^2 + 1$ *on* $[-2,\ 2]$.

6.5 Lagrange's Mean Value Theorem

Geometrically, Rolle's Theorem states that if a function is continuous on a closed and bounded interval $[a,\ b]$ and differentiablre on open and bounded interval $(a,\ b)$ such that at the end points a and b of $[a, b]$ the values of $f(x)$ are same. Then, there exists at least one point $c \in (a,\ b)$ such that the tangent at $(c,\ f\left(c\right))$ is parallel to the x-axis (i. e. $f'(x) = 0$). But a natural question arises, what if the value of f at the end points of interval a and b are not the same, then still we get some point between a and b such that the tangent at that point is parallel to x-axis? The answer is partially YES. In this case, we still get a tangent line at $(c,\ f\left(c\right))$ which is parallel to the line joining the end points $(a,\ f\left(a\right))$ and $(b,\ f\left(b\right))$.
This result is explained by Lagrange's Mean Value (L.M.V.) Theorem. We will also show that the Rolle's Theorem is a special case of Lagrange's Mean Value Theorem.

Theorem 6.4 (Lagrange's Mean Value Theorem)**.** *Let f be a real-valued function defined on $[a,\ b]$ satisfying the following properties*

1. *f is continuous on* $[a,\ b]$;

2. *f is differentiable on* $(a,\ b)$;

Then, there exists at least one $c \in (a,\ b)$ such that

$$f'\left(c\right) = \frac{\left(f\left(b\right) - f\left(a\right)\right)}{b-a}$$

or

$$f\left(b\right) = f\left(a\right) + f'\left(c\right)\left(b-a\right).$$

Proof. Suppose f is a function defined on $[a,\ b]$ such that f is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$.
Let us define another function F as

$$F\left(x\right) = f\left(x\right) + Ax \text{ on } [a,\ b]$$

where, A is a constant chosen such that $F(a) = F(b)$. That is, we have

$$f(b) + Ab = f(a) + Aa$$

or,

$$A = \frac{(f(b) - f(a))}{a - b} \tag{6.2}$$

Since the given function f is continuous and differentiable on $[a,\ b]$ and $(a,\ b)$ respectively, therefore F is also continuous on $[a,\ b]$ and differentiable on $(a,\ b)$. Also, by the construction of F, we have

$$F(a) = F(b)$$

Hence all the conditions of Rolle's Theorem hold for F, thus there exists at least one $c \in (a,\ b)$ such that

$$F'(c) = 0.$$

That is,

$$f'(c) + A = 0 \ \ \Rightarrow f'(c) = \ -A$$

Thus, from 6.2, we have

$$f'(c) = \frac{(f(b) - f(a))}{(b - a)}$$

or,

$$f(b) = f(a) + f'(c)(b - a)$$

Hence the proof. □

Lagrange's Mean Value Theorem is also known as *Frist Mean Value Theorem* of differential calculus.

Corollary 6.2. Suppose, in addition to the hypothesis of Lagrange's Mean Value Theorem, we have one more condition, $f(b) = f(a)$. Then by the conclusion of the Lagrange's Mean Value Theorem, we get a $c \in (a,\ b)$ such that

$$f'(c) = \frac{(f(b) - f(a))}{b - a} = 0.$$

Which leads to Rolle's Theorem. Thus, Rolle's Theorem is a special case of Lagrange's Mean Value Theorem.

Here, a natural question arises: why the name of Mean Value Theorem? What is the Mean Value stands for in the above theorem?
Here, we can see that $f(a)$ and $f(b)$ are the initial and the final value of the function f. Thus, $f(b) - f(a)$ is nothing but the total change in the value of f when the x-cooridante varies from a to b. That is, when x-coordinate changes from a to b, f moves from $f(a)$ to $f(b)$. Therefore, the average value (also known as mean value) of the change in the interval $[a, b]$ is $\frac{(f(b)-f(a))}{b-a}$.

The Lagrange's Mean Value Theorem asserts that average (mean) rate of change of the function f coincide with the derivative of f at some $c \in (a,\ b)$ (i. e., the instantaneous rate of change at c.)
In **physical terms,** the Lagrange's Mean Value Theorem says that the average velocity of a moving body during an interval of time, that is, $\frac{(f(b)-f(a))}{b-a}$ is coincide with the instantaneous velocity of the body at some moment $c \in (a,\ b)$.

Geometrical Interpretation of Lagrange's Mean Value Theorem

Geometrically, Lagrange's Mean Value Theorem states that there exists at least one point c lying between a and b such that the tangent line at $(c,\ f\,(c))$ on the curve $y = f\,(x)$, is parallel to the secant line joining the points $(a,\ f\,(a))$ and $(b,\ f\,(b))$, that is, their slope coincides, if f is continuous and differentiable between a and b. Hence, we have

$$\text{Slope of tangent at } (c,\ f\,(c)) = \text{slope the of secant line}$$

That is,

$$(f'\,(c)) = \left(\frac{f\,(b) - f\,(a)}{b - a}\right)$$

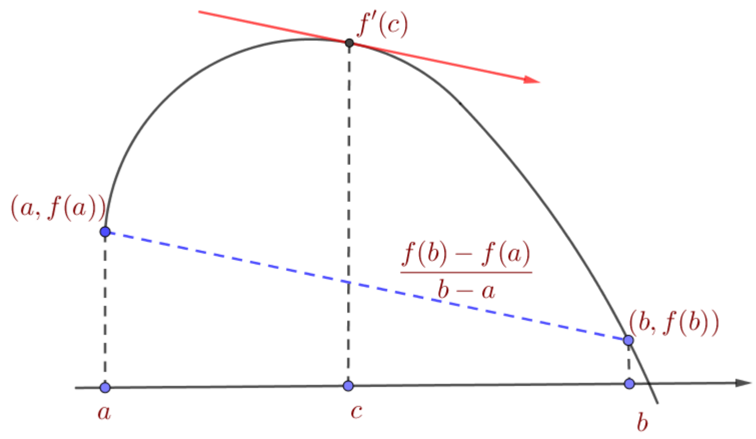

Figure 6.7: Lagrnage's Mean value Theorem

Example 6.10. Verify Lagrange's Mean Value Theorem for the function $f\,(x) = \cos x$ defined on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Solution. Consider the function $f\,(x) = \cos x$ defined on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Being a trigonometric function, $f\,(x) = \cos x$ is continuous on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ also it is differentiable on $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $f'(x) = -\sin x$.
Since both the conditions of Lagrange's Mean Value Theorem are satisfied therefore, there must exist some $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that

$$\begin{aligned} f'\,(c) &= \frac{\left(f\left(\frac{\pi}{4}\right) - f\left(-\frac{\pi}{4}\right)\right)}{\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)} \\ &= \frac{\cos(\frac{\pi}{4}) - \cos(\frac{-\pi}{4})}{\frac{\pi}{2}} \\ &= \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{\frac{\pi}{2}} \\ &= 0. \end{aligned}$$

Thus, we have

$$-\sin(c) = 0 \tag{6.3}$$

Therefore, We have $c = 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, such that $f'(c) = -\sin c = 0$. Hence Lagrange's Mean Value Theorem is verified.

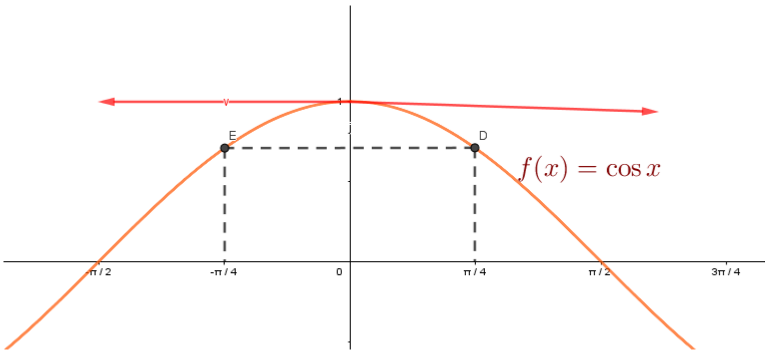

Figure 6.8: $y = \cos x$

Example 6.11. Verify the Lagrange's Mean Value Theorem for the function $f(x) = x^3 - 6x^2 - 2$ in the interval $[0,\ 2]$.

Solution. Consider the function $f(x) = x^3 - 6x^2 - 2$ on the interval $[0,\ 2]$. Here, f is continuous and differentiable, being a polynomial. Therefore, the conditions of the L.M.V. Theorem are satisfied. Thus, there exists some $c \in (0,\ 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{-18 + 2}{2} = -8.$$

That is,

$$3c^2 - 12c = -8$$

$$\text{Or,} \qquad 3c^2 - 12c + 8 = 0$$

Thus, we have

$$c = 2 \pm \frac{2}{3}\sqrt{3}.$$

Therefore, there exists $c = 2 - \frac{2}{\sqrt{3}} \in (0, 2)$ such that $f'(c) = 0$. Hence, the L. M. V. Theorem is verified.

Example 6.12. Find the points on the curve $y = \ln x$, where the tangent is parallel to the chord joining the points $(1,\ 0)$ and $(e,\ 1)$.

Solution. Consider the function $f(x) = \ln x$ on $[1,\ e]$. The function $y = \ln x$ is continuous on $[1,\ e]$ and differentiable on $(1,\ e)$ as well. Also, the given points $(1,\ e)$ and $(e,\ 1)$ both lie on the curve $y = \ln x$.

Therefore, the conditions of Lagrange's Mean Value Theorem, are satisfied, there exists some $c \in (1,\ e)$ such that

$$f'(c) = \frac{(f(b) - f(a))}{b-a}$$

That is,

$$\frac{1}{c} = \frac{\ln e \ \ - \ln 1}{e-1} = \frac{1}{e-1}$$

Therefore, we get $c = e-1 \in (1,\ e)$. Hence the required point on the curve is $(e-1, \ln(e-1))$.

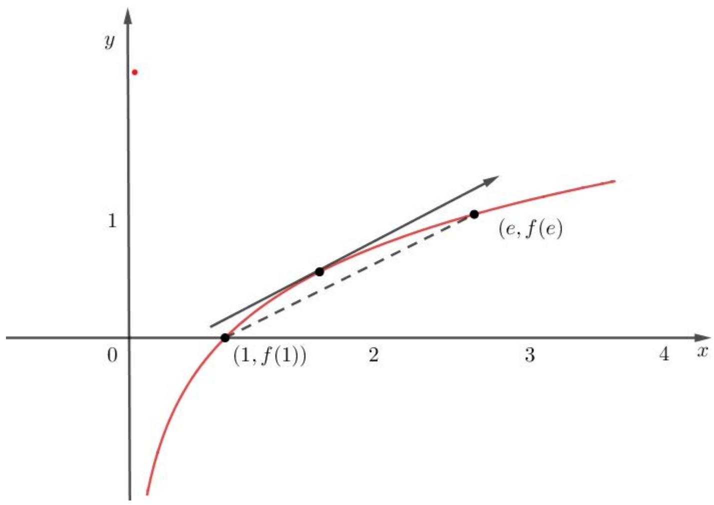

Figure 6.9: $y = \ln x$

Remark. Suppose, in the statement of Lagrange's Mean Value Theorem, we take $b = a+h$, then the conclusion of the Lagrange's Mean Value Theorem can be written as

$$f(a+h) = f(a) + hf'(c) \qquad \text{for some } c \in (a, a+h)$$

That is,

$$f(a+h) = f(a) + hf'(a+\theta h)$$

for some $\theta \in (0,\ 1)$.

Example 6.13. Let f be a real valued continuous function defined on $[a-h\ ,\ a+h]$ and differentiable on $(a-h,\ a+h)$. Then show that there exists a real number $0 < \theta < 1$ such that

$$f(a+h) - f(a-h) = h\left(f'(a+\theta h) + f'(a-\theta h)\right).$$

Solution. Consider a real-valued function $F(x) = f(a+hx) - f(a-hx)$ defined on $x \in [0,\ 1]$. Then, by the given hypothesis, F is continuous on $[0,\ 1]$ and differentiable on $(0,\ 1)$ (why?). Thus, by the Lagrange's Mean Value Theorem, there exists some $\theta \in (0,\ 1)$ such that

$$F(1) - F(0) = (1-0)\,F'(\theta)$$

Thus, we get

$$f(a+h) - f(a-h) = (f'(a+\theta h)(h) - f'(a-\theta h)(-h))$$
$$= h(f'(a+\theta h) + f'(a-\theta h))$$

Note. As we have already mentioned that Rolle's Theorem and Lagrange's Mean Value Theorem, both are existence theorems only. They just tell us that there always exists some $c \in (a,\ b)$ having some desirable property (that is, tangent at $(c,\ f(c))$ is parallel to x-axis or the secant line). But they do not tell us how to find that point, nor how many such points exist.

To illustrate this, consider the function $f(x) = x^3 - \sin x$ defined on $[0,\ 5\pi]$. Then, one can easily verify that f satisifes both the conditions of Lagrange's Mean Value Theorem, therefore there must exist some $c \in (0,\ 5\pi)$ such that

$$f'(c) = \frac{(f(5\pi) - f(0))}{5\pi - 0} = 25\pi^2$$
$$3c^2 - \cos(c) = 25\pi^2.$$

The Lagrange's Mean Value Theorem assures that the must exist a solution of the equation $3x^2 - \cos(x) - 25\pi^2 = 0$ in $(0,\ 5\pi)$. But it does not tell us, how to find the same.
In the following examples, we illustrate how the Lagrange's Mean Value Theorem is used for the monotonicity and inequalities of the functions.

Example 6.14. With the help of Lagrange's Mean Value Theorem, show that

$$|\sin x - \sin y| \leq |x - y| \text{ for all } x,\ y \in \mathbb{R}.$$

Hence, prove that

$$-x \leq \sin x \leq x \text{ for all } x \geq 0.$$

Solution. Consider the function $f(x) = \sin x$, which is continuous and differentiable for all $x \in \mathbb{R}$. Thus, by the Lagrange's Mean Value Theorem, we get some $c \in (x,\ y)$ such that

$$f(x) - f(y) = f'(c)(x - y)$$

that is,

$$\sin x - \sin y = \cos c\ (x - y)$$

$$\Rightarrow \quad |\sin x - \sin y| = |\cos c\ (x - y)|$$
$$\leq |x - y| \text{ as } |\cos c| \leq 1.$$

Hence, we get

$$|\sin x - \sin y| \leq |x - y| \text{ for all } x,\ y \in \mathbb{R}.$$

Now, taking $y = 0$, we get

$$|\sin x - \sin(0)| = |\sin x| \leq |x - 0|$$

Thus, we have

$$|\sin x| \leq |x| \text{ for all } x \in \mathbb{R}.$$

If $x \geq 0$, we get $-x \leq \sin x \leq x$.

Example 6.15. Using the Lagrange's Mean Value Theorem, prove that

$$\frac{x}{1+x} < \log(1+x) \ < x, \text{ for } x > 0.$$

Hence, show that

$$0 < \frac{1}{\log(1+x)} \ - \frac{1}{x} < 1, \text{ for } x > 0.$$

Solution. Consider the function $f(x) = \log(1+x)$ for $x \in [0, x]$, $x > 0$. Since, the logarithmic function is continuous on $[0, x]$ and differentiable on $(0, x)$, thus, by the conclusion of Lagrange's Mean Value Theorem, there exists some $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\log(1+x)}{x}$$

That is,

$$\frac{1}{1+c} = \frac{\log(1+x)}{x}$$

$$\Rightarrow \qquad \log(1+x) \ = \frac{x}{1+c} \tag{6.4}$$

Now, for $0 < c < x$, we have $1 + c < 1 + x$ thus, we have

$$\frac{x}{1+x} < \frac{x}{1+c} \tag{6.5}$$

Also, we have $\frac{x}{1+c} < x$. By using (6.5), we get

$$\frac{x}{(1+x)} < \frac{x}{1+c} < x, \text{ that is, } \frac{x}{1+x} < \log(1+x) \ < x.$$

$$\Rightarrow \qquad \frac{1+x}{x} > \frac{1}{\log(1+x)} > \frac{1}{x}$$

$$\Rightarrow \qquad 1 + \frac{1}{x} > \log(1+x) \ > \frac{1}{x}$$

Hence

$$1 > \frac{1}{\log(1+x)} - \frac{1}{x} > 1 \text{ for } x > 0.$$

Example 6.16. By using Lagrange's Mean Value Theorem, find an approximate value of $\sqrt{50}$.

Solution. Consider the function

$$f(x) = \sqrt{x} \text{ for } x \geq 0.$$

Then, it can be easily verified that f is continuous on $[a, b]$ and differentiable on (a, b) for some $0 \leq a < b$.
Thus, by the Lagrange's Mean Value Theorem, we get some $c \in (a, b)$ satisfying

$$f'(c)(b - a) = f(b) - f(a)$$

That is,

$$\sqrt{b} - \sqrt{a} = \frac{(b-a)}{2\sqrt{c}}.$$

In particular, if we take $b = 50$ and $a = 49$, then

$$sqrt50 - 7 = \frac{1}{2\sqrt{c}} \quad \text{for some } c \in (49,\ 50) \tag{6.6}$$

Now, $49 < c < 50$, thus $49 < c < 64$.

$$7 < \sqrt{c} < 8$$

$$\frac{1}{7} > \frac{1}{\sqrt{c}} > \frac{1}{8} \Rightarrow \frac{1}{14} > \frac{1}{2\sqrt{c}} > \frac{1}{16} \qquad \text{by using } (6.6)$$

Hence,

$$\frac{1}{14} > \sqrt{50} - 7 > \frac{1}{16} \Rightarrow \frac{1}{14} + 7 > \sqrt{50} > \frac{1}{16} + 7$$

Thus, we get $7.0714 > \sqrt{50} > 7.0625$, that is, $\sqrt{50}$ is approximately equal to $\frac{7.0625+7.0714}{2} = 7.06695$

Example 6.17. Show that for $0 < a < b$,

$$\frac{b-a}{1+b^2} < \tan^{-1}b \ - \tan^{-1}a \ < \frac{b-a}{1+a^2}$$

Hence, deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} \ < \frac{\pi}{4} + \frac{1}{6}.$$

Solution. Consider the function

$$f\,(x) = \tan^{-1}x \quad \text{on } [a,\ b]$$

Then, f is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$. Then, by the Lagrange's Mean Value Theorem, there exists some $c \in (a,\ b)$ such that

$$f\,(b) - f\,(a) = f'\,(c)\,(b-a)$$

that is,

$$\tan^{-1}b \ - \tan^{-1}a \ = \frac{1}{1+c^2}(b-a)$$

Now, as $a < c < b$

$$\Rightarrow \qquad \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$Rightarrow \qquad \frac{b-a}{1+b^2} < \tan^{-1}b \ -\tan^{-1}a \ < \frac{b-a}{1+a^2} \tag{6.7}$$

By taking, $a = 1$ and $b = \frac{4}{3}$, in (6.7), we get

$$\frac{\frac{4}{3} - 1}{1 + \left(\frac{4}{3}\right)^2} < \tan^{-1}\frac{4}{3} \; - \tan^{-1}1 \; < \frac{\frac{4}{3} - 1}{1 + 1^2}$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} \; - \frac{\pi}{4} < \frac{1}{6}.$$

Hence, we get

$$\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} \; < \frac{1}{6} + \frac{\pi}{4}.$$

In-text Exercise 6.2. *1. Verify the Lagrange's Mean Value Theorem for the following functions :*

(*a*) $f(x) = x^2$ *on* $[0,\ 5]$;

(*b*) $g(x) = x^3$ *on* $[-1,\ 1]$;

2. *At what point, the tangent to the curve* $y = x^n$ *for* $n \in \mathbb{N}$ *is parallel to the chord from from the point* $(0,\ 0)$ *to* $(1,\ 1)$.

3. *Verify the Lagrange's Mean Value Theorem for* $f(x) = (x-1)(x-2)(x-3)$ *on* $[0,\ 4]$.

6.6 Some applications of Mean Value Theorem

In this section, we will illustrate several applications of the mean value theorem. Prior to that, we will delve into a notable property of functions, referred to as "monotonicity", which encompasses the concepts of "increasing" and "decreasing". Through the process of differentiation, we can determine whether a function is increasing, decreasing, or neither. We will present the subsequent analytical definitions for a monotonic function.

Definition 6.2. Let I be an interval in the domain of a real valued function f. Then f is said to be

1. *increasing* on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ for all $x_1, x_2 \in I$

2. *decreasing* on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ for all $x_1, x_2 \in I$

3. *constant* on I if $f(x) = 0$ for all $x \in I$

Next, we will introduce the first derivative test for discerning the behavior of functions in terms of their monotonicity.

Theorem 6.5. *Let f be a continuous function on $[a, b]$ and differentiable on (a, b). Then*

1. *f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$;*

2. *f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$;*

3. *f is a constant function in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.*

The Mean Value Theorem has various simple yet important applications which reveal the behaviour of a function f and its derivative f'. Here, we mention few of them.
Let us consider a real-valued function f which is differentiable on a closed and bounded interval $[a,\ b]$. Let x_1 and x_2 be any two points in the interval $[a,\ b]$ such that $b > x_2 > x_1 > a$. Then, by applying the Lagrange's Mean Value Theorem on the interval $[x_1, x_2]$, we get some $c \in (x_1, x_2)$ such that

$$f\left(x_2\right) - f\left(x_1\right) = (x_2 - x_1)f'(c) \tag{6.8}$$

1. Let $f'\left(x\right) = 0$ throughout the interval $(a,\ b)$, then from the equation 6.8, we get

$$f\left(x_2\right) - f\left(x_1\right) = 0 \Rightarrow f\left(x_2\right) = f(x_1)$$

   Since $x_1$and x_2 are chosen as arbitrary numbers. Hence f is a constant function.

2. Let f and F be two real-valued functions such that they both have same derivative for every value of $x \in (a,\ b)$. then they differ by a constant only.

   Let us consider a function
$$\phi\left(x\right) = f\left(x\right) - F(x)$$
   Then, we have
$$\phi'\left(x\right) = f'\left(x\right) - F'\left(x\right) = 0$$
   Hence, form the part 1, we get $f\left(x\right) - F\left(x\right)$ is a constant.

3. Let $f'\left(x\right) > 0$ for every value of $x \in (a,\ b)$. Then form the equation (6.8) , we get

$$f\left(x_2\right) - f\left(x_1\right) = \left(x_2 - x_1\right) f'\left(c\right) > 0 \qquad \text{for } x_2 > x_1$$

   That is,
$$f\left(x_2\right) - f\left(x_1\right) > 0 \qquad \text{for } x_2 > x_1$$
   Hence f is an increasing function.

   Similarly, one can prove that if $f'\left(x\right) < 0$ for all $x \in [a,\ b]$ then f is a decreasing function on $[a, b]$.

In-text Exercise 6.3. *1. Show that $f\left(x\right) = x^3 - 3x^2 + 4x + 2$ is monotonically increasing function.*

2. Find the intervals in which the function

$$f\left(x\right) = 2x^3 - 15x^2 + 36x + 1$$

is increasing or decreasing.

6.7 Summary

In this chapter, we have discussed the following points:

1. **Rolle's Theorem**
   If f is a real-valued function defined on a closed and bounded interval $[a,\ b]$ satisfying the following properties:

   (a) f is continuous on $[a, b]$;

   (b) f is differentiable on $(a,\ b)$;

   (c) $f\,(a) = f(b)$

   Then there always exists at least a point $c \in (a, b)$ such that

   $$f'\,(c) = 0.$$

2. Geometrical interpretation of Rolle's Theorem.

3. Applications of Rolle's Theorem such as to locate the roots of an equation.

4. **Lagrange's Mean Value Theorem**
   Suppose f is a real-valued function defined on $[a,\ b]$ satisfying the following properties

   (a) f is continuous on $[a,\ b]$;

   (b) f is differentiable on $(a,\ b)$;

   Then, there exists some $c \in (a,\ b)$ such that

   $$f'\,(c) = \frac{(f\,(b) - f\,(a))}{b-a}$$

   That is,

   $$f\,(b) = f\,(a) + f'\,(c)\,(b-a)\,.$$

5. Geometrical interpretation of Lagrange's Mean Value Theorem.

6. Applications of Lagrange's Mean Value Theorem, such as

   (a) Suppose $f'(x) = 0$ for all $x \in (a, b)$ then f is constant.

   (b) Suppose $f'(x) > 0$ for all $x \in (a, b)$ then f is an increasing function.

   (c) Suppose $f'(x) < 0$ for all $x \in (a, b)$ then f is a decreasing function.

6.8 Self Assessment Exercises

1. Prove that between any two real roots of $e^x \text{sin} x = 1$ there is at least one real root of $e^x \text{cos} x = -1$.

2. Using $f(x) = (x-4)\ln x$, prove that the equation $x \ln x + x = 4$ is satisfied by at least one value of x in $(1, 4)$.

3. Consider a function $f(x) = (x-2)(x-1)$. Prove that $f(-1) = f(4)$. Also, find a point $c \in (-1, 4)$ such that $f'(c) = 0$.

4. Verify the Lagrange's Mean Value Theorem for the following functions :

   (a) $h(x) = \text{sin}\pi x$ on $[0, 4]$;

   (b) $j(x) = 2x^3 + 4$ on $[0, 2]$.

5. At what point is the tangent to the curve $y = x^n$ for $n \in \mathbb{N}$ parallel to the chord joining the points $(0, 0)$ and $(2, 2^n)$.

6. Use the Lagrange's Mean Value Theorem to prove that

   (a) $|\text{sin} ax - \text{sin} bx| \leq |a-b|\,|x|$ for $x \neq 0$;

   (b) $|\text{cos} ax - \text{cos} bx| \leq |a-b|\,|x|$ for $x \neq 0$;

   (c) $0 < \frac{1}{x}\log\frac{e^x-1}{x} < 1$.

7. Prove that $e^x > (1+x)$ for all $x (\neq 0) \in \mathbb{R}$.

8. Prove that
$$\frac{2}{\pi} < \frac{\text{sin} x}{x} < 1 \qquad \text{for } 0 < x < \frac{\pi}{2}.$$

9. Prove that
$$\frac{\text{tan} x}{x} < \frac{x}{\text{sin} x} \qquad \text{for } 0 < x < \frac{\pi}{2}$$

6.9 Solutions to In-text Exercises

Exercise 6.1

1. (a) Here, we have $f(x) = (x-a)^m(x-b)^n$ defined on $[a, b]$, where $m, n \in \mathbb{N}$. Being a polynomial f is continuous and differentiable. Also, $f(a) = f(b) = 0$. Thus, all the conditions of Rolle's Theorem hold. Thus, there must exist some $c \in (a, b)$ such that
$$f'(c) = 0$$
that is,
$$m(c-a)^{m-1}(c-b)^n + n(c-a)^m(c-b)^{n-1} = 0$$
Therefore, we have $c = \frac{mb+na}{m+n} \in (a, b)$ such that $f'(c) = 0$.

(*b*) Here, we have $f(x) = \log \frac{x^2+ab}{(a+b)x}$ defined on $[a, b\,]$. Then, f is continuous and differentiable (Verify it!). Also, $f(a) = f(b) = 0$. Thus, all the conditions of Rolle's Theorem holds true. Thus, there must exist some $c \in (a, b)$ such that

$$f'(c) = 0$$

that is,

$$\frac{(a+b)x}{(x^2+ab)} \frac{(a+b)x.2x - (x^2+ab)(a+b)}{(a+b)^2x^2} = 0$$

Therefore, we have $c = \sqrt{ab} \in (a, b)$ such that $f'(c) = 0$.

2. (*a*) Consider $f(x) = \frac{\sin x}{e^x}$ defined on $[0, \pi]$. Then, f is continuous and differentiable (Verify it!). Also, $f(0) = f(\pi) = 0$. Thus, all the conditions of Rolle's Theorem holds true. Thus, there must exist some $c \in (0, \pi)$ such that

$$f'(c) = 0$$

that is,

$$\frac{e^c(\sin c - \cos c)}{e^{2c}} = 0$$

Therefore, we have $c = \pi/4 \in (0, \pi)$ such that $f'(c) = 0$.

(*b*) Here, we have $f(x) = e^(x)(\sin x - \cos x)$ defined on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. Here f is continuous and differentiable. Also, $f(\pi/4) = f(5\pi/4)$. Thus, by the Rolle's Theorem, there must exist some $c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that

$$f'(c) = 0$$

that is

$$e^c(\sin c - \cos x) + e^c(\cos c + \sin c) = 0$$

Hence, we have $2e^c \sin c = 0$, thus, we have $\sin c = 0$. Hence $c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

3. (*a*) Here $f(x) = |x - 2|$ is not differentiable at $x = 2$. Thus the hypothesis of Rolle's Theorem is not satisfied. Hence, Rolle's Theorem does not hold.

(*b*) Do Yourself!

(*c*) Here, we have $h(0) = 0 \neq h(1) = 1$. Thus, Rolle's Theorem is not applicable.

(*d*) Here $j(0) = 0$ but $j(1) = 2$. Hence the conditions of Rolle's Theorem are not satisfied,

(*e*) Rolle's Theorem is satisfied and we have $k'(c) = 2c = 0$ at $c = 0 \in (-2, 2)$.

Exercise 6.2

1. (*a*) Consider $f(x) = x^2$ defined on $[0, 5]$. Being polynomial, f is continuous and differentiable. Therefore, the conditions of Lagrange's Mean Value Theorem are satisfied. Thus, by Lagrange's Mean Value Theorem, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is,

$$2c = \frac{25 - 0}{5 - 0} = 5$$

Hence, we have $c = \frac{5}{2} \in (0, 5)$. Thus, Lagrange's Mean value Theorem is satisfied.

(b) Here, we have $g(x) = x^3$ defined on $[-1, 1]$. It is continuous as well as differentiable for all $x \in [-1, 1]$.
Now,

$$g'(c) = \frac{g(b) - g(a)}{b - a}$$

That is,

$$3c^2 = \frac{1 + (-1)}{1 - (-1)} = 1$$

Hence, we have $c = \sqrt{\frac{1}{3}} \in (-1, 1)$. Thus, Lagrange's Mean value Theorem is verified.

2. Consider $y = x^n$ defined on $[0, 1]$. The given curve is continuous and differentiable. Thus, there must exist some $c \in (0, 1)$ such that the tangent at $(c, f(c))$ is parallel to the chord joining by the point $(0, 0)$ and $(1, 1)$, where the point $c \in (0, 1)$ is

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 1$$

That is, we have $nx^{n-1} = 1 \Rightarrow x = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}$.

3. Do it yourself ! ($c = 0.845$ and $3.154 \in (0, 4)$)

Exercise 6.3

1. Consider $f(x) = x^3 - 3x^2 + 4x + 2$. We have, $f'(x) = 3x^2 - 6x + 4 = 3(x^2 - 2x + 1 - 1) + 4 = 3(x - 1)^2 + 1 > 0$ for all $x \in \mathbb{R}$. Thus, $f(x)$ is monotonically increasing function.

2. Consider $f(x) = 2x^3 - 15x^2 + 36x + 1$. After differentiating with respect to x, we get

$$f'(x) = 6x^2 - 30x + 36$$

For the critical points, we have $f'(x) = 0$, that is, $6x^2 - 30x + 36 = 0$, or, we can say $6(x - 2)(x - 3) = 0$.

Hence, $x = 2, 3$ are critical points of $f(x)$.

Here, one can observe that $f'(x) > 0$ whenever $x < 2$ and $x > 3$, $f'(x) < 0$ for $2 < x < 3$. Hence, the given function is increasing on the interval $(-\infty, 2) \cup (3, \infty)$ and decreasing on the interval $(2, 3)$.

Lesson - 7

Expansion of Functions

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Structure

7.1 Learning Objectives

- To learn the Cauchy's Mean value Theorem and its applications;
- To understand the geometrical interpretation of Cauchy's Mean Value Theorem;
- To learn Taylor's theorem with illustrations;
- To learn and implement the expansions of functions by using Taylor's series and Maclaurin's series.

7.2 Introduction

In the previous lessons, we have studied the Mean value Theorems (namely Rolle's Theorem and Lagrange's Mean Value Theorem) and their applications in calculus. In this lesson, we further continue our study about the differentiable functions in the light of Mean value Theorems. We also pen down some applications of the Mean value Theorems in calculus.

We will introduce two more generalizations of the Mean value Theorem namely Cauchy's Mean value Theorem and Taylor's Theorem. In general Mean value Theorem talks about the existence of some point in the domain of the function which relates the value of function with its first derivative. In this lesson, we will study Cauchy's Mean value Theorem, which deals with two continuous functions and their first order derivatives simultaneously. While, Taylor's Theorem provides some relationship between the values of a function with its higher order derivatives.

7.3 Cauchy's Mean Value Theorem

In this section, we study a generalization of the Mean value Theorem namely Cauchy's Mean value Theorem. We also discuss some applications for the Cauchy's Mean Value Theorem. We will also prove some results regarding arithmetic and geometric mean in view of the Cauchy's Mean value Theorem.

Theorem 7.1. *Let f and g be two real-valued functions defined on a closed and bounded interval $[a,b]$ such that*

(i) *f and g both are continuous on $[a,b]$;*

(ii) *f and g both are differentiable on (a,b);*

(iii) *$g'(x) \neq 0$ for all $x \in [a,b]$.*

Then, there exists at least a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

that is,

$$(f(b)-f(a))\, g'(c) = (g(b)-g(a))\, f'(c)$$

Proof. Let f and g be two real-valued functions, defined on closed and bounded interval $[a,b]$, such that f and g both are continuous on $[a,b]$ and differentiable on (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$.

Now, consider a function F defined as

$$F(x) = f(x) + Ag(x)$$

where A is a constant chosen such that $F(a) = F(b)$. Hence, we have

$$f(a) + Ag(a) = f(b) + Ag(b)$$

that is,

$$A = -\frac{f(b)-f(a)}{g(a)-g(b)} \tag{7.1}$$

Here, one can observe that $g(a) \neq g(b)$. Let if possible, let $g(b) = g(a)$. Then g being continuous on $[a,b]$ and differentiable on (a,b), g satisfies all the conditions of Rolle's

Theorem, thus by the conclusion of Rolle's Theorem, there must exist some $c \in (a, b)$ such that

$$g'(c) = 0$$

which leads to a contradiction as $g'(x) \neq 0$ for all $x \in (a, b)$.

Since f and g both are continuous on $[a, b]$, therefore F is also continuous on $[a, b]$. Similarly, F is differentiable on (a, b) and by the choice of A, we have $F(a) = F(b)$. Thus, all the condition of Rolle's Theorem hold for the function F and hence there must exist some $c \in (a, b)$ such that

$$\begin{aligned} F'(c) &= 0 \\ \text{i. e. } f'(c) + Ag'(c) &= 0 \\ \Rightarrow \frac{f'(c)}{g'(c)} &= -A \\ \text{or } \frac{f'(c)}{g'(c)} &= \frac{f(b) - f(a)}{g(b) - g(a)}, \text{ by using (7.1)} \end{aligned}$$

Hence the proof. □

The Cauchy's Mean value Theorem is also known as the *Second Mean Value Theorem* of differential calculus.

Remark. If we consider $g(x) = x$ for all $x \in [a, b]$. Then, the real-valued function g is continuous and differentiable on $[a, b]$ (being a polynomial). Also, $g'(x) = 1 \neq 0$ for all $x \in [a, b]$. Thus, by the Cauchy's Mean Value Theorem, we get

$$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{for some } c \in (a, b)$$

Thus, one can conclude that Lagrange's Mean Value Theorem is a particular case of Cauchy's Mean Value Theorem.

Alternate Form of Cauchy's Mean value Theorem

Let f and g are two real-valued functions defined on closed and bounded interval $[a, a+h]$ for some $h > 0$. If

(i) f and g both are continuous on $[a, a + h]$;

(ii) f and g both are differentiable on $(a, a + h)$;

(iii) $g'(x) \neq 0$ for all $x \in (a, a + h)$

then, there exists a real number $\theta \in (0, 1)$ such that

$$\frac{f(a + h) - f(a)}{g(a + h) - g(a)} = \frac{f'(a + \theta h)}{g'(a + \theta h)}.$$

Geometrical Interpretation of Cauchy's Mean Value Theorem

In this section, we learn the significance of Cauchy's Mean Value Theorem geometrically.

Cauchy's Mean value Theorem claims that if, we have a parametric equation of a curve $x = g(t)$ and $y = f(t)$ for $a \leq t \leq b$ and suppose f and g both are continuous on $[a, b]$ and differentiable on (a, b). Then, for any $t \in (a, b)$, where $g'(t) \neq 0$, the slope of the curve at the point $(g(t), f(t))$ is

$$m = \frac{f'(t)}{g'(t)}$$

The Cauchy's Mean value Theorem says that under the given hypothesis, there must exist some $c \in (a, b)$ for which the slope of the tangent at the point $(g(c), f(c))$ coincides with the slope of the chord joining by the points $(g(b), f(b))$ and $(g(a), f(a))$ of the curve.

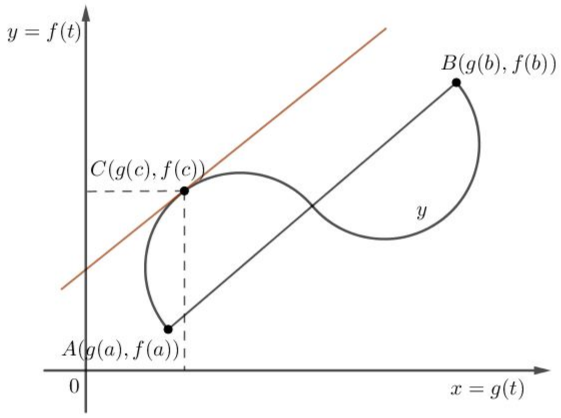

Figure 7.1: Cauchy's Mean Value Theorem

The Cauchy's Mean Value Theorem can also be concluded as follow:

If all the hypothesis of the Cauchy's Mean Value Theorem holds good then, one can re-write the conclusion as

$$\frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}} = \frac{f'(c)}{g'(c)}$$

that is,

$$\frac{\text{slope of the chord joining } (a, f(a)) \text{ and } (b, f(b))}{\text{slope of the chord joining } (a, g(a)) \text{ and } (b, g(b))} = \frac{\text{slope of the tangent at } (c, f(c))}{\text{slope of the tangent at } (c, g(c))}$$

Now, we will illustrate the Cauchy's Mean Value Theorem, with the help of some examples. We also discuss some applications as well.

Example 7.1. Verify the Cauchy's Mean Value theorem for the function

$$f(x) = x^4 \qquad \text{and} \qquad g(x) = x^2$$

on the interval $[1, 2]$.

Solution. Consider the functions $f(x) = x^4$ and $g(x) = x^2$ defined on $[1, 2]$. Then, being polynomial functions

(i) f and g both are continuous on $[1, 2]$;

(ii) f and g both are differentiable on $(1, 2)$.

Also, $g'(x) = 2x \neq 0$ for all $x \in (1, 2)$. All the conditions of Cauchy's Mean Value Theorem are satisfied, therefore there exist some $c \in (1, 2)$ such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

That is,

$$\frac{16 - 1}{4 - 1} = \frac{15}{3} = \frac{4c^3}{2c} = 2c^2$$

$$\Rightarrow 2c^2 = 5 \qquad \text{or } c = \pm\sqrt{\frac{5}{2}} = \pm 1.58$$

Here, we take $c = 1.58 \in (1, 2)$. Thus, the Cauchy's Mean Value Theorem is verified.

Example 7.2. Verify the Cauchy's Mean Value Theorem for the functions

$$f(x) = x^3 \qquad \text{and} \qquad g(x) = \arctan x$$

on the interval $[0, 1]$.

Solution. Consider the functions $f(x) = x^3$ and $g(x) = \arctan x$ defined on $[0, 1]$. Since f is a polynomial functions and g is an inverse trigonometric function, therefore Then, one can verify that

(i) f and g both are continuous on $[0, 1]$;

(ii) f and g both are differentiable on $(0, 1)$.

Also, $g'(x) = \frac{1}{1+x^2} \neq 0$ for all $x \in (0, 1)$. Thus, all the hypothesis of Cauchy's Mean Value Theorem is verified. Therefore there exists some $c \in (0, 1)$ such that

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)}$$

That is,

$$\frac{1 - 0}{\frac{\pi}{4} - 0} = \frac{3c^2}{\frac{1}{1+c^2}}$$

$$\frac{4}{\pi} = \frac{1 + c^2}{c^2} \Rightarrow 12c^2 = \pi + \pi c^2$$

$$c^2 = \frac{\pi}{12 - \pi}$$

. Thus, $c = \pm\sqrt{\frac{\pi}{12-\pi}} = \pm 0.60$.
Hence, we get $c = 0.60 \in (0, 1)$. Thus, the Cauchy's Mean value Theorem is verified.

Example 7.3. Show that

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$$

where $0 < \alpha < \theta < \beta < \pi/2$.

Solution. Here, let us define two functions $f(x) = \sin x$ and $g(x) = \cos x$ on the interval $[\alpha, \beta]$, where $0 < \alpha < \beta < \pi/2$. Both the functions f and g are continuous, being trigonometric function on the interval $[\alpha, \beta]$ and differentiable on (α, β). Also, $g'(x) = -\sin x \neq 0$ for all $x \in (\alpha, \beta)$.

Hence, by the Cauchy's Mean Value Theorem, there exists at least one real number $\theta \in (\alpha, \beta)$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

That is,

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta$$

Thus, we have

$$\frac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} = \cot\theta, \text{ where } 0 < \alpha < \theta < \beta < \pi/2.$$

Example 7.4. By using the Cauchy's Mean Value theorem, illustrate that the arithmetic mean and geometric mean of two numbers a and b always lie between a and b.

Solution. We illustrate as following:

(i) Let us consider two functions $f(x) = x^2$ and $g(x) = x$ defined on closed and bounded interval $[a, b]$, (and without any loss of generality, we assume that $a < b$).

Here, both the functions f and g are continuous and differentiable on $[a, b]$. Also, $g'(x) = 1 (\neq 0)$ for all x. Hence, by the Cauchy's Mean Value Theorem, there must exist some number c between a and b such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

that is, we have

$$\frac{b^2 - a^2}{b - a} = \frac{2c}{1} \Rightarrow b + a = 2c$$

Hence, we get some $c \in (a, b)$ such that

$$c = \frac{a + b}{2} \in (a, b).$$

Now, we will illustrate that geometric mean of two positive real numbers a and b, $\sqrt{ab}$ is always lie between them, that is,

$$a < \sqrt{ab} < b$$

(ii) Consider two functions $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$ defined on $[a, b]$, where $0 < a < b$.

One can easily verified that f and g both are continuous function on $[a, b]$. Also, $f'(x) = \frac{1}{2\sqrt{x}}$ and $g'(x) = \frac{-1}{2x\sqrt{x}} (\neq 0)$ for all $x > 0$. Thus, by the Cauchy's Mean value Theorem, there must exist some $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

that is, we have

$$\frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}} \Rightarrow \left(\sqrt{b} - \sqrt{a}\right)\left(\frac{\sqrt{ab}}{\sqrt{a} - \sqrt{b}}\right) = -c$$

Thus, we have

$$c = \sqrt{ab} \in (a, b)$$

Hence the result.

In-text Exercise 7.1. *1. Verify the Cauchy's Mean Value Theorem for:*

(i) $f(x) = x(x-1)(x-2)$ *and* $g(x) = x(x-2)(x-3)$ *on the interval* $\left[0, \frac{1}{2}\right]$*;*

(ii) $f(x) = \sin x$ *and* $g(x) = \cos x$ *on the interval* $\left[-\frac{\pi}{2}, 0\right]$.

7.4 Sequences and Series

In this section, we will discuss about more analytical concepts of real numbers – the sequences and infinite series. Here, we will discuss about the limiting process, the most fundamental concept of analysis. The concepts of convergence of a sequence and a series are helpful to discuss the Taylor's and Maclaurin's Series in Section 7.5.

Sequence

Definition 7.1. A *sequence* of real numbers or a *real sequence* is a bijective function from the set of natural numbers $\mathbb{N}$ to the set of real numbers $\mathbb{R}$.

Since we will discuss only real sequences here, we use the word sequence to denote a real sequence.

Usually, a sequence $f\colon \mathbb{N} \to \mathbb{R}$ is often expressed by

$$(f_n) = (f_1, f_2, \ldots, f_n, \ldots)$$

or

$$(x_n) = (x_1, x_2, \ldots, x_n, \ldots), \qquad x_n = f_n$$

x_n is termed as the nth term of the sequence (x_n).

Example 7.5. 1. $((-1)^n)$ is the sequence whose n^{th} term is $(-1)^n$ and the sequence is $(-1, 1, -1, \ldots,)$;

2. $\left(\frac{1}{n}\right)$ is the sequence whose n^{th} term is $1/n$ and the sequence is $(1, 1/2, 1/3, \ldots,)$;

Apart from these, there exists some known sequences such as

1. **Constant sequence**

   A sequence in which each term is equal to some $a \in \mathbb{R}$, for example, the sequence $(a, a, a, \ldots)$ is called a *constant sequence*;

2. **Recursive Sequence**

   A sequence may be defined with the help of some recursive formula. In this case, the term of the sequence depends on its previous terms.

   For example, the sequence consisting of odd natural numbers can be defined as

$$\begin{aligned} x_1 &= 1 \\ x_{n+1} &= x_n + 2, \qquad \text{for all } n \geq 1 \end{aligned}$$

now, consider the following sequences

$$\left(\frac{1}{n^2}\right), (n), (1 + (-1)^n)$$

If we closely observe the behaviour of these sequences for sufficiently large values of n. we can say that, as n increases, the terms of first sequence decreases gradually and move towards zero. Whereas, in the second sequence, for large values of n, the terms of this sequence approaches to ∞. On the other hand, the terms of these sequences oscillates between 0 and 2.

Out of these three sequences, the first one is a convergent sequence with zero as its limit, while the other two are the divergent one.

We now provide a formal definition of the convergence of the sequence.

Definition 7.2. Let $(x_n) = (x_1, x_2, x_3, \ldots,)$ be a real sequence. A real number x is said to be a *limit* of (x_n) if for every $\varepsilon > 0$, there always exists a natural number $m \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \qquad \text{for all } n \geq m$$

If a sequence (x_n) has a limit x, then we say that (x_n) is *convergent* and it converges to x. Symbolically, we write

$$\lim_{n\to\infty} (x_n) = x \quad \text{or} \quad \lim(x_n) = x \quad \text{or} \quad x_n \to x$$

If (x_n) does not have finite limit, then we say that the sequence is *non-convergent* or *divergent*.

Example 7.6. The sequence $\left(\frac{n}{n+1}\right)$, that is $(1/2, 2/3, 3/4, \ldots)$ converges to the limit 1.

Remark. 1. The natural number m chosen in definition 7.2 of limit depends on ε. Which means that for different choices of ε, we shall get different value of m;

2. For given $\varepsilon > 0$, one can get more than one m, that is, the choice of m is not unique;

3. Limit of a sequence, if it exists, is always unique.

By the inspection, one can easily determine the limit of the following sequences:

1. The sequence (a^n), for $0 < a < 1$ is a convergent sequence, which converge to 0;

2. sequence $\left(\frac{1+2+3+...+n}{n^2}\right) = \left(\frac{n+1}{2n}\right) = \left(\frac{1+\frac{1}{n}}{2}\right)$ converges to $\frac{1}{2}$;

3. (na^n) for $0 < a < 1$ converges to 0.

Following are some examples of convergent sequences, which help us to check whether a sequence is convergent or divergent.

Definition 7.3. **(Bounded Sequence)** A sequence (x_n) is said to be *bounded* if there exists some $M > 0$ such that

$$|x_n| \leq M \qquad \text{for all } n$$

Example 7.7. One can easily observe that $((-1)^n)$, $\left(\frac{1}{n^2}\right)$ are examples of bounded sequences as we have

$$|(-1)^n| \leq 1, \left|\frac{1}{n^2}\right| < 1, \qquad \text{for all } n \in \mathbb{N}$$

whereas (n) and $(n^2 + 1)$ etc. are unbounded sequences.

Now, We provide some properties of convergent sequence in terms of boundedness.

Theorem 7.2. *Every convergent sequence is bounded.*

Proof. (Left as exercise.) □

Remark. The converse of the above result does not hold in general. For example, consider the sequence $((-1)^n)$, it is a bounded sequence with the bound $M = 1$, but not convergent.

From the above theorem, one can conclude that if a sequence is not bounded, then it must be divergent.

Example 7.8. Verify whether the sequence (n^2) is convergent.

Solution. Let if possible, (n^2) is a convergent sequence. Therefore, it must be bounded. Hence, there must exists some $M > 0$ such that for all $n \in \mathbb{N}$

$$n^2 < M$$

$$\text{or} \qquad n < \sqrt{M}$$

which leads to a contradiction as for each real number x (in particular for $\sqrt{m}$) there exists $n \in \mathbb{N}$ such that $n > x$. Therefore, (n^2) can not be bounded and hence can not be convergent.

Cauchy Sequences

Although boundedness provides us a necessary condition for the convergence of a sequence. But it does not provide us the sufficient conditions, there are of examples of sequences, which are bounded but not convergent. Below, we discuss an important criterion, known as *Cauchy Convergence Criterion*, which will provide both necessary and sufficient conditions for the convergence of real-sequences.

Definition 7.4. A sequence (x_n) is said to be a *Cauchy Sequence* if for given $\varepsilon > 0$, there exists a natural number N such that

$$|x_n - x_m| < \varepsilon \qquad \text{for all } n, m \geq N$$

In simple words, we can say that the difference between the terms become very small as the sequence progresses. That is, if we drop a few finite numbers of terms from the beginning, then the distance between any two of the remaining terms will be arbitrarily small.

It can be easily derived that

1. Every convergent sequence is a Cauchy sequence;

2. Every Cauchy sequence is bounded.

That is, if a sequence is not bounded, then it is not a Cauchy sequence. Similarly, a sequence which is not Cauchy, is not convergent.

Thus, for the set of real numbers $\mathbb{R}$, we can give the following conditions:

Theorem 7.3. ***(Cauchy's Criterion of Convergence)***
A sequence (x_n) is convergent if and only if it is a Cauchy Sequence.

That is, a sequence (x_n) is convergent if and only if for given any $\varepsilon > 0$, there exists a natural number N such that

$$|x_n - x_m| < \varepsilon \qquad \text{for all } n, m \geq N$$

In-text Exercise 7.2. *1. Find the limit of the following sequences.*

(a) $\left(\sqrt{n+1} - \sqrt{n}\right)$

(b) $\left(n^{1/n}\right)$

(c) $\left(\frac{1}{\sqrt{n!}}\right)$

(d) (2^n)

(e) $((-1)^n n)$

Infinite Series

In this section, we will learn the concept of infinite series of real n umbers. We will study the convergence and divergence of an infinite series.

Let (x_n) be a real sequence. Then, the sum

$$\sum_{n=1}^{\infty} = x_1 + x_2 + x_3 + \ldots + x_n + \ldots$$

is called an *infinite series*.

The sum of first n terms of the infinite series $\sum_{n=1}^{\infty} x_n$ is denoted by S_n. The sequence (S_n) is known as the *sequence of partial sum* of the series $\sum_{n=1}^{\infty} x_n$.

Consider a series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$$

also known as geometric series, is an example of an infinite series.

Let (S_n) be the sequence of partial sum of the above mentioned geometric series and let S_n denote the sum of first n terms. Then, we have

$$S_n = 2 - \left(\frac{1}{2}\right)^{n-1}$$

Here, one can observe that for larger values of n, $\left(\frac{1}{2}\right)^{n-1}$ becomes very small. In other words, by choosing a sufficiently large n, the term $\left(\frac{1}{2}\right)^{n-1}$ can be made as small as we please. Thus, for this sufficiently large n, the sequence of partial sum (S_n) approaches to 2 . A series, in which (S_n) tends to a finite limit is called a *convergent series* and the sum of the series is $\lim_{n\to\infty} (S_n)$..

Definition 7.5. An infinite series $\sum x_n$ is said to be *convergent* if its sequence of partial sum (S_n) is convergent. Also, let

$$\lim_{n\to\infty} (S_n) = S$$

then S is called the sum of the series $\sum_{n=1}^{\infty} x_n$ and we write

$$\sum_{n=1}^{\infty} x_n = S.$$

Since, we link the convergence of the infinite series with the convergence of a sequence, therefore the convergence of the series can be easily studied in the light of the convergence of a sequence.

Now, consider the following infinite series

$$1 + 2 + 2^2 + 2^3 + \ldots$$

$$1 - 1 + 1 - 1 + 1 - \ldots$$

$$1 - 3 + 5 - 7 + 9 - 11 + \ldots$$

In the first series, $S_n = 2^n - 1$ and as n increases, S_n also increases and by choosing sufficiently large n, S_n can be made to very large and can exceed any given number, however large that number may be chosen. Therefore S_n tends to infinity as n tends to infinity and therefore the series (1) is divergent.

But in the second series, its sequence of partial sum (S_n) is equal to either 0 or 1, (depends on n, either is odd or even). Similarly, in the third series, S_n is alternatively positive and negative, while S_n increases numerically with n. Such series, where the sum oscillates from one value to another is known as *oscillatory series.*

Note. Rules of algebra, like addition, multiplication etc. can not be applied to the infinite series also.

Consider an infinite series

$$S = 1 + 2 + 4 + 8 + 16 + \ldots$$

after multiplying by 2, we get

$$2S = \quad 2 + 4 + 8 + 16 + \ldots$$

after subtracting the above two equations, we get

$$S = -1$$

which is absurd, since all the terms in S are positive. It happens because S is not a convergent series. Multiplication and addition in series holds only, when they are convergent.

Example 7.9. Discuss the convergence of the series

$$a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} + \ldots$$

where, $a \in \mathbb{R}$ is a constant and $x \in \mathbb{R}$.

Solution. The given series is a geometric series with the first term a and the common ratio x. Thus, the sum of first n terms is

$$S_n = \frac{a(1 - x^n)}{1 - x}$$

Here, the convergence of (S_n) depends on x only. If $|x| < 1$, then $|x|^n$ decreases continuously and tend to 0, as n tends to infinity. Hence

$$S = \lim_{n \to \infty} (S_n) = \frac{a}{1 - x}$$

Thus, the given series is convergent, if $|x| < 1$.

If $|x| > 1$, then $|x|^n$ continuously increases with n. Hence the series diverges for $x > 1$.

For $x = 1$, the series becomes

$$a + a + a + \ldots +$$

which is divergent as $S_n = na$ and increases rapidly with n.

For $x = -1$, we have

$$a - a + a - a + \ldots$$

Here, S_n is alternatively a and 0, therefore, the series oscillates..

Example 7.10. Find the sum of the first n terms of the following series and discuss their convergence:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
2. $\displaystyle\sum_{n=1}^{\infty} n^2$

Solution. 1. Let $\displaystyle\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Then

$$x_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Thus, we have $x_1 = 1 - \frac{1}{2}$, $x_2 = \frac{1}{2} - \frac{1}{3}, \ldots x_n = \frac{1}{n} - \frac{1}{n-1}$.

Let S_n be the sequence of partial sum of the given series, then we have

$$S_n = x_1 + x_2 + x_3 + \ldots + x_n = 1 - \frac{1}{n+1}$$

Therefore, $\displaystyle\lim_{n\to\infty} (S_n) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$

Hence, the sequence of partial sum (S_n) converges to 1. Thus, we have $\displaystyle\sum_{n=1}^{\infty} x_n = 1.$

2. Here, $x_n = n^2$ and the sum of first n terms is

$$S_n = x_1 + x_2 + x_3 + \ldots + x_n = 1 + 2^2 + 3^2 + \ldots + n^2$$

Therefore, $S_n = \frac{n(n+1)(2n+1)}{6}$ and $\displaystyle\lim_{n\to\infty} (S_n) = \infty$. Thus, the sequence of partial sum (S_n) diverges to ∞, therefore $\displaystyle\sum_{n=1}^{\infty} x_n$ also diverges to ∞.

7.5 Taylor's Theorem

Taylor's Theorem gives us a method to approximate a given function by a polynomial. Approximating a function by a polynomial is a very useful in calculus. It is because of the fact that polynomials are one of the simplest kind of functions for which differentiation, integration etc. are easy to compute. Taylor's Theorem gives us the tool to look for the suitable polynomial for a given function also to describe the error involved in the approximation.

As mentioned before, Taylor's Theorem is a more general form of Mean value Theorem. It is applicable when the function is continuously differentiable.

Theorem 7.4. ***(Taylor's Theorem)*** *Let f be a real-valued function defined on $[a, b]$ such that*

(i) *the nth derivative, $f^{(n)}$ is continuous on $[a, b]$;*

(ii) *the nth derivative, $f^{(n)}$ is differentiable on (a, b), That is, $f^{(n+1)}$ exists on (a, b).*

Then, we have

$$\begin{aligned} f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots + \\ \frac{(b-a)^r}{r!}f^r(a) + \ldots + \frac{(b-a)^n}{(n)!}f^n(a) \\ + \frac{(b-a)^{n+1}}{(n+1)!}f^{n+1}(a + \theta(b-a)) \end{aligned} \tag{7.2}$$

for some $\theta \in (0, 1)$.

Proof. Let f be a real-valued function defined on $[a, b]$ such that its nth derivatives, $f', f'', \ldots, f^{n-1}$ exist on $[a, b]$. Then consider a function

$$\begin{aligned} F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \ldots \\ - \frac{(b-x)^r}{r!}f^r(x) - \ldots \\ - \frac{(b-x)^n}{n!}f^n(x) - \frac{(b-x)^{n+1}}{(n+1)!}Q \end{aligned}$$

where Q is a constant chosen such that $F(a) = F(b)$. Since $F(b) = 0$, therefore, we have $F(a) = 0$.

$$\begin{aligned} 0 = & f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \ldots \\ & - \frac{(b-a)^r}{r!}f^r(a) - \ldots \\ & - \frac{(b-a)^n}{n!}f^n(a) - \frac{(b-a)^{n+1}}{(n+1)!}Q \end{aligned} \tag{7.3}$$

One can observe that $F(x)$ consists of a finite number of terms and f^n exists therefore $F'(x)$ exists for all $x \in [a,b]$ and by the construction $F(a) = F(b)$, thus all the conditions of Rolle's Theorem are satisfied by $F(x)$ on $[a, b]$. Therefore, there exists some $c \in (a, b)$ such that

$$F'(c) = 0$$

that is, we have

$$-f'(c) + f'(c) - (b-c)f''(c) + (b-c)f''(c) - \frac{(b-c)^2}{a}f'''(c) + \dots$$
$$-\frac{(b-c)^n}{n!}f^{(n)}(c) + \frac{(b-c)^n}{n!}Q = 0.$$

Thus, we have

$$\frac{(b-c)^n}{n!} - \big(Q - f^{(n+1)}(c)\big) = 0.$$

that is,

$$\Rightarrow Q = f^{(n+1)}(c) \tag{7.4}$$

Therefore, from equations (7.3) and (7.4), we get

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots +$$
$$\frac{(b-a)^r}{r!}f^r(a) + \dots + \frac{(b-a)^n}{n!}f^{(n)}(a)$$
$$+ \frac{(b-a)^{(n+1)}}{(n+1)!}f^{(n+1)}(c)$$

or,

$$f(b) = f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}+\dots+\frac{(b-a)^n}{n!}f^{(n)}(a)+\frac{(b-a)^{(n+1)}}{(n+1)!}f^{(n+1)}(a+\theta(b-a)) \text{ for some } 0 < \theta \tag{7.5}$$

Hence, the proof.

□

Remark. Usually, Taylor's Theorem is quoted in the following forms:

(i) By replacing b by x in Equation (7.5), we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots +$$
$$\frac{(x-a)^r}{r!}f^r(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$
$$+ \frac{(x-a)^{(n+1)}}{(n+1)!}f^{(n+1)}(a+\theta(x-a)), \qquad 0 < \theta < 1 \tag{7.6}$$

(ii) By replacing b by $x+h$ and a by x in Equation 7.5, we get

$$\begin{aligned} f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \\ \frac{h^r}{r!}f^r(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) \\ + \frac{h^{(n+1)}}{(n+1)!}f^{(n+1)}(x+\theta h), \qquad 0 < \theta < 1 \end{aligned} \tag{7.7}$$

Remark. We have from equation (7.6) that

$$f(x) = T_n(x) + R_n(x)$$

where

$$T_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) \tag{7.8}$$

and

$$R_n(x) = \frac{(x-a)^{(n+1)}}{(n+1)!}f^{(n+1)}(a+\theta(x-a)). \tag{7.9}$$

The polynomial $T_n(x)$ in equation (7.8) is called the *Taylor's Polynomial* of degree n and $R_n(x)$ in equation (7.8) is known as the *remainder term*. If we approximate $f(x)$ by $T_n(x)$, then $R_n(x) = f(x) - T_n(x)$ is the error term in this approximation.

That is, if all the conditions of Taylor's Theorem holds good for a function f on $[a,b]$, then one can decompose the same into two parts, (i) Polynomial (ii) and some remainder.

Later, we will study that for a sufficiently large n, whenever the remainder term converge to 0, then the given function $f(x)$ converges to $T_n(x)$.

In the following examples, we illustrate the above stated Taylor's Theorem.

Example 7.11. Find the Taylor's Polynomial of degree 3 for the function $f(x) = \sin x$ about $x = 0$.

Solution. Let us consider the function $f(x) = \sin x$, which is continuous and differentiable for all $x \in \mathbb{R}$. Then, we have

$$\begin{aligned} f(x) &= \sin x &\Rightarrow\quad f(0) &= \sin(0) = 0; \\ f'(x) &= \cos x &\Rightarrow\quad f'(0) &= \cos(0) = 1; \\ f''(x) &= -\sin x &\Rightarrow\quad f''(0) &= -\sin(0) = 0; \\ f'''(x) &= -\cos x &\Rightarrow\quad f'''(0) &= -\cos(0) = -1; \end{aligned}$$

Therefore, we have

$$\begin{aligned} T_3(x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(x) \\ &= 0 + x + 0\frac{x^2}{2!} + \frac{-1}{3!}x^3 \\ &= x - \frac{1}{6}x^3 \end{aligned}$$

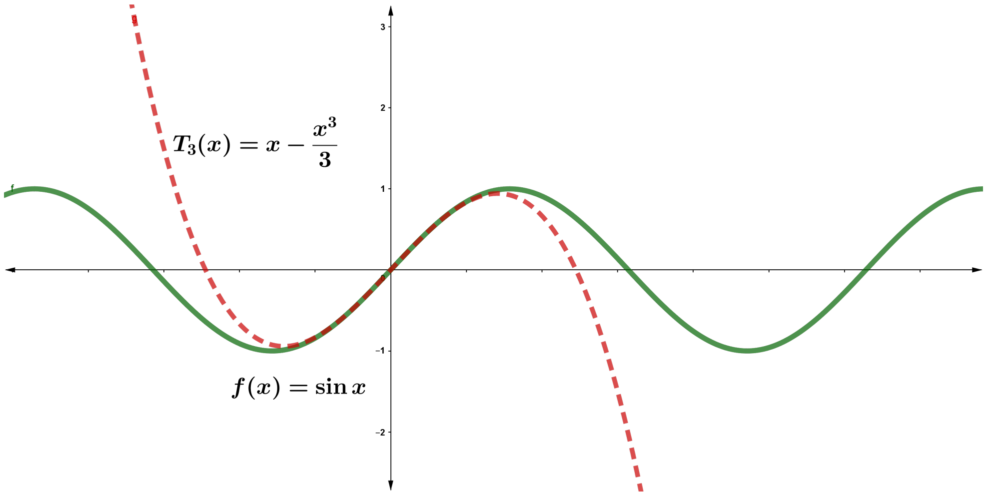

Figure 7.2: Graph of $\sin x$ and its Taylor Polynomial of degree 3 about 0

Example 7.12. Find the Taylor's Polynomial of degree 3 of the function

$$f(x) = 3 + 5x^2 - 4x^3 + x^4$$

about $x = 1$.

Solution. Consider the function

$$f(x) = 3 + 5x^2 - 4x^3 + x^4,$$

which and continuous and differentiable for all x, also we have $f(1) = 5$. Now, we have

$$\begin{array}{rcllrcl} f(x) &=& 3 + 5x^2 - 4x^3 + x^4 & \Rightarrow & f(1) &=& 5; \\ f'(x) &=& 10x - 12x^2 + 4x^3 & \Rightarrow & f'(1) &=& 2; \\ f''(x) &=& 10 - 24x + 12x^2 & \Rightarrow & f''(1) &=& -2; \\ f'''(x) &=& -24 + 24x & \Rightarrow & f'''(1) &=& 0; \end{array}$$

Thus, the Taylor's Polynomial of degree 3 of the given function is

$$\begin{aligned} T_3(x) &= f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) \\ &= 5 + 2(x-1) - (x-1)^2. \end{aligned}$$

In-text Exercise 7.3. *1. Find the Taylor's polynomial of degree 4 of the following functions*

(i) $f(x) = \cos x$ *about* $x = 0$;

(ii) $f(x) = e^x$ *about* $x = 0$;

Lagrange's and Cauchy's Forms of Remainders

In the Taylor's theorem discussed and illustrated above the remainder term

$$R_n(x) = \frac{(x-a)^{(n+1)}}{(n+1)!} f^{(n+1)}(a + \theta(x-a))$$

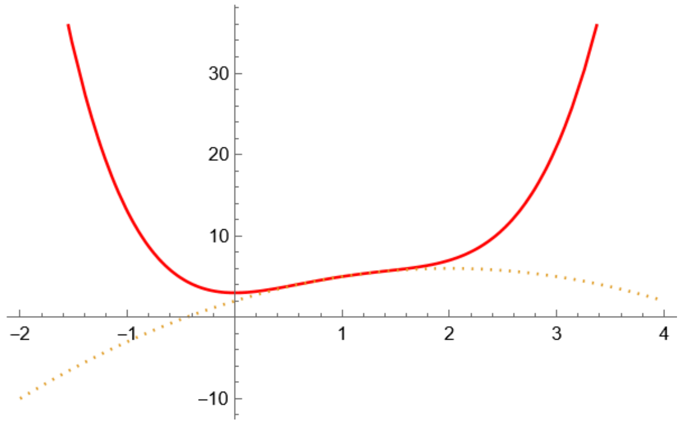

Figure 7.3: Graph of f and its Taylor Polynomial of degree 3

given by (7.9) depends on $n \in \mathbb{N}$. It is also known as ***Lagrange form of remainder***. There is another form of the remainder term $R_n(x)$ as well, known as ***Cauchy's form of remainder***, It is presented in the following theorem.

Theorem 7.5. *Let f be real-valued function defined on $[x, x+h]$ such that*

(i) *the nth derivative $f^{(n)}$ is continuous on $[x, x+h]$;*

(ii) *the nth derivative $f^{(n)}$ is differentiable on $(x, x+h)$*

then, there exists a real number θ between 0 and 1 such that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{(n+1)}}{n!}(1-\theta)^n f^{(n+1)}(x+\theta h) \tag{7.10}$$

That is,

$$f(x+h) = T_n(x) + R_n(x), \text{ where}$$

$$T_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

and

$$R_n(x) = \frac{h^{(n+1)}}{n!}(1-\theta)^n f^{(n+1)}(x+\theta h), \; 0 < \theta < 1$$

is the Cauchy's form of remainder.

Maclaurin's Theorem

Suppose f is a real-valued function, satisfies all the conditions of Taylor's Theorem in the given interval $[a, a+h]$ and suppose $x \in [a, a+h]$. Then f also satisfies all the conditions in the interval $[a, x]$.

By replacing $a + h$ by x in the expression stated above, we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{(n+1)}}{(n+1)!}f^{(n+1)}(a+\theta(x-a))$$

(Lagrange's form of Remainder)

or

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{(n+1)}}{n!}(1-\theta)^n f^{(n)}(a+\theta(x-a))$$

(Cauchy's form of Remainder)

On taking $a = 0$ in the above equations, we get:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{(n+1)}}{(n+1)!}f^{(n+1)}(\theta x), \qquad 0 < \theta < 1$$

and

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{(n+1)}}{n!}(1-\theta)^n f^{(n+1)}(\theta x), \qquad 0 < \theta < 1 \tag{7.11}$$

As stated earlier, if the remainder term $R_n(x)$ in the expression for $f(x)$ using Lagrange's form of remainder or Cauchy's form of remainder tend to 0 as n tends to ∞, the $f(x)$ converges to polynomial $T_n(x)$. Therefore, for both cases, we can write

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) \tag{7.12}$$

or

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots \tag{7.13}$$

This is known as *Maclaurin's* Series expansion of $f(x)$

7.6 Some Standard Expansion

In this section, we use Maclaurin's Series, to discuss the power series expansion of some functions.

1. **Expansion for e^x**

   Let us consider

$$f(x) = e^x, \qquad x \in \mathbb{R}$$

Then we have $f^{(n)}(x) = e^x$ for all $n \in \mathbb{N}$, thus f possesses derivatives of all order for all values of $x \in \mathbb{R}$. Therefore, from Maclaurin's Theorem, with Lagrange's Form of remainder, we have

$$\begin{aligned} f(x) =& T_n(x) + R_n(x) \\ =& f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \\ & \frac{x^n}{n!}f^{(n)}(0) + R_n(x) \end{aligned}$$

where,

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x) = \frac{x^{n+1}}{(n+1)!}e^{\theta x}$$

As, we know that $\frac{x^n}{n!} \to 0$, when $n \to \infty$ for all x. Therefore,

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!}e^{\theta x} = 0 \tag{7.14}$$

Hence, we have the Maclaurin's series expansion of $f(x)$ as

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

for all $x \in \mathbb{R}$.

2. **Expansion for** $\cos x$

We have $f(x) = \cos x$,

$$f^{(n)}(x) = \cos\left(\frac{n\pi}{2} + x\right) \qquad \text{for all } n \in \mathbb{N}$$

Now, from the Maclaurin's Theorem with Lagrange's Form of Remainder, we get

$$\begin{aligned} f(x) &= T_n(x) + R_n(x) \\ &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \\ & \frac{x^n}{n!}f^{(n)}(0) + R_n(x) \end{aligned}$$

where $R_n(x) = \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x)$ for some $\theta \in (0, 1)$. Thus,

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\sin\left((n+1)\frac{\pi}{2} + \theta x\right)$$

Therefore,

$$|R_n(x)| = \left|\frac{x^{n+1}}{(n+1)!}\sin\left(n+1\frac{\pi}{2} + \theta x\right)\right| \leq \left|\frac{x^{n+1}}{(n+1)!}\right|\left|\sin\left((n+1)\frac{\pi}{2} + \theta x\right)\right| \leq \left|\frac{x^{n+1}}{(n+1)!}\right|$$

Therefore, when $n \to \infty$, $R_n(x) \to 0$ for all value of $x \in \mathbb{R}$. Hence, all conditions of Maclaurin's series holds good. Hence, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

where, $f^{(n)}(0) = \cos\left(\frac{n\pi}{2}\right)$. Thus, we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

for all $x \in \mathbb{R}$.

3. **Expansion for** $\log(1+x)$

   Let us consider $f(x) = \log(1+x)$ for $x \in (-1, 1]$.

   We have

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \qquad \text{for all } n \in \mathbb{N} \text{ and for all } x > -1$$

   Therefore, f possesses derivatives of every order for $x > -1$. Hence, by the Maclaurin's Theorem, we have

$$\begin{aligned} f(x) &= T_n(x) + R_n(x) \\ &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + R_n(x) \end{aligned}$$

   where $R_n(x)$ may be either Lagrange's form of Remainder or Cauchy's form of Remainder. In the following, we consider two cases:

$0 < x \leq 1$ Consider the Lagrange's form of Remainder for $x \in (-1, 0)$. That is,

$$\begin{aligned} R_n(x) &= \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x) \\ &= \frac{(-1)^n}{n+1}\left(\frac{x}{1+\theta x}\right)^{n+1} \end{aligned}$$

for some $\theta \in (0, 1)$.
Since, we have $x \in (0, 1)$ and $\theta \in (0, 1)$, thus we have $0 < \frac{x}{1+\theta x} < 1$. That is

$$|R_n(x)| < \frac{1}{n+1} \qquad \text{and} \qquad \frac{1}{n+1} \to 0 \text{ as } n \to \infty$$

Therefore, we have $R_n(x) \to 0$.

$-1 < x < 0$ In this case $\left|\frac{x}{1+\theta x}\right|$ need not be less than 1. Hence, its not advisable to use Lagrange's form of remainder, therefore, one can use Cauchy's form of remainder in this case. Consider

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(\theta x)}{n!}(1-\theta)^n x^{n+1} \\ &= \frac{(-1)^n n!}{(1+\theta x)^{n+1}} \frac{(1-\theta)^n}{n!} x^{n+1} \\ &= (-1)^n \left(\frac{1-\theta}{1+\theta x}\right)^n \frac{1}{1+\theta x} x^{n+1} \qquad \text{for some } \theta \in (0,1) \end{aligned}$$

Now, we have $x \in (-1, 0)$ and $\theta \in (0, 1)$, therefore $0 < 1-\theta < 1+\theta x$. Hence

$$0 < \frac{1-\theta}{1+\theta x} < 1$$

Similarly, $|x| < 1$, hence $x^n \to 0$ as $n \to \infty$. Thus, for sufficiently large n, we have $R_n(x) \to 0$.

Thus, from the above two cases, we get, whenever $-1 < x \leq 1$, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

where $f(x) = \log(1+x)$

$$f(0) = 0, f'(0) = 1, f''(0) = -1, f'''(0) = (-1)^2 2!, \dots$$

Thus, the Maclaurin's Series expansion of $\log(1+x)$ is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \qquad \text{for } -1 < x \leq 1$$

Note. While using Maclaurin's Series expansion, one can use any form of remainder.

4. **Expansion for** $(1+x)^m$

Suppose $m \in \mathbb{N}$, then f possesses derivatives for all n and for all $x \in \mathbb{R}$. Also, $f^{(n)}(x) = 0$ whenever $n > m$. Thus, we consider the following two cases:

- m **is a positive integer**

  In this case, f possesses derivatives of all order for each $x \in \mathbb{R}$ such that

  $$f'(x) = m(1+x)^{m-1},\ f''(x) = m(m-1)(1+x)^{m-2}, \dots, f^{(m)}(x) = m!$$

  and $f^{(n)}(x) = 0$ for all $n > m$, thus for sufficiently large n, we have $R_n(x) = 0$. Hence, we have

  $$\begin{aligned} f(x) &= T_n(x) + R_n(x) \\ &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^m}{m!}f^{(m)}(0) + 0 \\ &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \dots + x^m \end{aligned}$$

  which is the required Maclaurin's Series.

- $m \notin \mathbb{N}$

  In this case, we have

$$f^{(n)}(x) = m(m-1)(m-2)\dots(m-n+1)(1+x)^{(m-n)} \qquad \text{for } x \neq 1$$

Here, we consider Cauchy's form of Remainder, that is

$$\begin{aligned} R_n(x) &= \frac{(1-\theta)^n x^{n+1}}{n!} f^{(n+1)}(\theta x) \\ &= \frac{m(m-1)(m-2)\dots(m-n)(1-\theta)^n}{n!}(1+\theta x)^{m-n+1} x^{n+1} \\ &= \frac{m(m-1)(m-2)\dots(m-n)}{n!}\left(\frac{1-\theta}{1+\theta x}\right)^n (1+\theta x)^{m-1} x^{n+1} \end{aligned}$$

Now, we will show that for the sufficiently large n, that is, as $n \to \infty$, we have $R_n(x) \to \infty$.
For $|x| < 1$ and $\theta \in (0, 1)$, we have

$$0 < \frac{1-\theta}{1+\theta x} < 1 \Rightarrow \left(\frac{1-\theta}{1+\theta x}\right)^n \to 0 \text{ as } n \to \infty$$

Also $(1+\theta x)^{m-1}$ is finite and

$$\frac{m(m-1)\dots(m-n)}{n!} \to 0\text{as } n \to \infty$$

Thus, for the sufficiently large n, we have $R_n(x) \to 0$ and we have

$$\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots \text{where } f(x) = (1+x)^m \\ f(0) &= 1, f'(0) = m, f''(0) = m(m-1), f'''(0) = m(m-1)(m-2), \dots \end{aligned}$$

Thus, the Maclaurin's Series expansion of $(1+x)^m$ is

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{3}x^3 + \dots \qquad \text{for } |x| < 1$$

In-text Exercise 7.4. 1. *Assuming the validity of Taylor's Expansion, show that*

$$\sin x = \frac{1}{\sqrt{2}}\left(1 + \left(x - \frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} + \dots\right)$$

2. *Assuming the possibility of expansion, expand* $\tan^{-1} x$ *as far as the term containing* x^5.

7.7 Summary

In this lesson, we have discusses the following topics:

1. If f and g are real-valued function defined on a closed and bounded interval $[a,\ b]$ such that

   (a) f and g both are continuous on $[a, b]$;

   (b) f and g both are differentiable on $(a,\ b)$;

   (c) $g'(x) \neq 0$ for all $x \in (a, b)$.

   Then there always exists at least a point $c \in (a, b)$ such that

   $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

   It is known as *Cauchy's Mean Value Theorem.*

2. Lagrange's Mean value Theorem is a particular case of Cauchy's Mean Value Theorem.

3. Cauchy's Mean Value Theorem can be restated as follow:

   If f and g are real-valued function defined on a closed and bounded interval $[a, a+h]$ for some $h > 0$, such that

   (a) f and g both are continuous on $[a, a + h]$;

   (b) f and g both are differentiable on $(a,\ a + h)$;

   (c) $g'(x) \neq 0$ for all $x \in (a, a + h)$

   then, there exists some $\theta \in (0, 1)$ such that

   $$\frac{f(a + h) - f(a)}{g(a + h) - g(a)} = \frac{f'(a + \theta h)}{g'(a + \theta h)}$$

4. **Taylor's Theorem**

   Let f be a real-valued function defined on $[a, b]$ such that

   (i) the nth derivative, $f^{(n)}$ is continuous on $[a, b]$;

   (ii) the nth derivative, $f^{(n)}$ is differentiable on (a, b). That is, $f^{(n+1)}$ exists on $(a, b$.

   Then,

   $$\begin{aligned} f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots + \\ \frac{(b-a)^r}{r!}f^r(a) + \ldots + \frac{(b-a)^n}{n!}f^{(n)}(a) \\ + \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(a + \theta(b-a)) \end{aligned}$$

   for some value of $\theta \in (0, 1)$.

5. Taylor's Theorem is also quoted in the following forms:

   (i) By replacing b by x in the Taylor's Theorem (Finite Form), we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^r(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta(x-a))$$

   (ii) But on replacing $b-a$ with h and put x for a in Taylor's Theorem, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^r}{r!}f^r(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x)\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x+\theta h)$$

6. In the finite form of Taylor's Theorem, the n^{th} Taylor's Polynomial is defined as

$$T_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a)$$

   and the remainder term is denoted by $R_n(x)$, which is defined as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta(x-a)).$$

   Thus, Taylor's Theorem can be represented as

$$f(x) = T_n(x) + R_n(x) \tag{7.15}$$

7. The Lagrange's form of Remainder is defined as

$$R_n(x) = \frac{h^n}{n!}f^n(a+\theta h)$$

8. The Cauchy's form of Remainder is defined

$$R_n(x) = \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^n(a+\theta h)$$

9. When, we define the taylor's series about $x = 0$, then Taylor's Series reduces to Maclaurin's Series.

10. Maclaurin's Series expansion of some standard functions

    (a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

(c) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

(d) $(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \frac{m(m-1)(m-2)}{3}x^3 + \dots \qquad$ for $|x| < 1$

11. A sequence of real numbers is a bijective function from the set of natural numbers $\mathbb{N}$ to the set of real numbers $\mathbb{R}$;

12. A sequence (x_n) is said to be convergent to x, if for given $\varepsilon > 0$, there exists a natural number $m \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \qquad \text{for all } n \geq m$$

13. Every convergent sequence is bounded.

14. A sequence (x_n) is said to be *Cauchy* if for given $\varepsilon > 0$, there exists a natural number N such that

$$|x_n - x_m| < \varepsilon \qquad \text{for all } n, m \geq N$$

15. A sequence is convergent if and only if it is Cauchy.

16. An infinite series $\sum x_n$ is convergent if its sequence of partial sum (S_n) is convergent, where S_n is the sum of first n terms of $\sum x_n$.

7.8 Self-Assessment Exercises

1. Verify the Cauchy's Mean Value Theorem for:

   $f(x) = 1/x^2$ and $g(x) = 1/x$ on $[a, b]$ for $a > 0$;

2. If we consider, $f(x) = 1/x^2$ and $g(x) = 1/x$ in the Cauchy's Mean Value Theorem, then show that c is the harmonic mean of the a and b.

3. Show that for $x \neq 0$

$$1 - \frac{x^2}{2} < \cos x.$$

4. Find the Taylor's polynomial of degree 4 for the function

   (a) $f(x) = 3 + 5x + 5x^2 - 10x^3 + 12x^4$ about $x = 2$;

   (b) $f(x) = \tan x$ about the origin;

   (c) $f(x) = \ln x$ about $x = -1$;

5. Assuming the validity of expansion, prove that

   (a) $\log \sec x = \frac{x^2}{2!} + \frac{x^4}{12} + \dots$

   (b) $\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \dots$

(c) $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{12} + \ldots$

6. Show that the sequence (x_n) where $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$ is convergent.

7. Show that the sequence $x_n = (-1)^n + 1$ oscillates finitely.

8. Show that
$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0.$$

9. Show that
$$\lim_{n\to\infty} \left(\frac{3n!}{(n!)^3}\right)^{1/n} = 0.$$

10. Show that $x_n = (-1)^n/n$ is a Cauchy sequence.

11. Show that
$$x_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$$
is not a Cauchy sequence.

7.9 Solutions to In-text Exercises

Exercise 7.1

(i) Consider $f(x) = x(x-1)(x-2)$ and $g(x) = x(x-2)(x-3)$ defined on $\left[0, \frac{1}{2}\right]$. One can easily observe that f and g both are continuous and differentiable on $\left[0, \frac{1}{2}\right]$. Also, $g'(x) = 3x^2 - 10x + 6 \neq 0$ for all $x \in \left[0, \frac{1}{2}\right]$. Thus, all the conditions of Cauchy's Mean value Theorem hold. Thus, there exists some $c \in \left(0, \frac{1}{2}\right)$ such that
$$\frac{f\left(\frac{1}{2}\right) - f(0)}{g\left(\frac{1}{2}\right) - g(0)} = \frac{f'(c)}{g'(c)}$$
that is,
$$\frac{\frac{3}{8}}{\frac{15}{8}} = \frac{3c^2 - 6c + 2}{3c^2 - 10c + 6}$$
After, simplification, we get $c = \frac{1}{6}\left(5 \pm \sqrt{13}\right)$. Thus $c = \frac{1}{6}\left(5 + \sqrt{13}\right) = 0.232 \in (0, 1/2)$. Thus, Cauchy's Mean value Theorem is satisfied.

(ii) Consider $f(x) = \sin x$ and $g(x) = \cos x$ defined on $\left[-\frac{\pi}{2}, 0\right]$. Here, being trigonometric functions, f and g both are continuous and differentiable on the given interval. Also $g'(x) = -\sin x$, which is non-zero in $\left(-\frac{\pi}{2}, 0\right)$. Hence, by the hypothesis of Cauchy's Mean Value Theorem, there must exist some $c \in \left(-\frac{\pi}{2}, 0\right)$ such that
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

that is,

$$\frac{\cos c}{-\sin c} = 1$$

Hence, we get $\tan c = -1$, that is $c = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, 0\right)$. Thus, the Cauchy's Mean Value Theorem is satisfied.

Exercise 7.2

1. 0.

2. 1.

3. 0.

4. sequence is unbounded and hence divergent.

5. sequence is oscillatory unbounded.

Exercise 7.3

(i) Consider the function $f(x) = \cos x$, which is continuous and differentiable for all $x \in \mathbb{R}$. Then, we have

$$\begin{array}{rclcrcl} f(x) &=& \cos x & \Rightarrow & f(0) &=& \cos(0) = 1; \\ f'(x) &=& -\sin x & \Rightarrow & f'(0) &=& -\sin(0) = 0; \\ f''(x) &=& -\cos x & \Rightarrow & f''(0) &=& -\cos(0) = -1; \\ f'''(x) &=& \sin x & \Rightarrow & f'''(0) &=& \sin(0) = 0; \\ f^{iv}(x) &=& \cos x & \Rightarrow & f^{iv}(0) &=& \cos(0) = 1; \end{array}$$

Therefore, we have

$$\begin{aligned} T_4(x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(x) + \frac{x^4}{4!}f^{iv}(x) \\ &= 1 + x.0 + (-1)\frac{x^2}{2!} + 0.\frac{x^3}{3!} + \frac{1}{4!}x^4 \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \end{aligned}$$

(ii) here $f(x) = \cos x$, which is continuous and differentiable for all $x \in \mathbb{R}$. Also, we have

$$f^n(x) = e^x \qquad x \in \mathbb{R}$$

and $f^n(0) = 1$ for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} T_4(x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(x) + \frac{x^4}{4!}f^{iv}(x) \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \end{aligned}$$

Exercise 7.4

(i) Let us consider

$$f(x) = \sin x = \sin\left(\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\right) = \sin(a+h)$$

where, $a = \frac{\pi}{4}$ and $h = x - \frac{\pi}{4}$

By assuming the validity of expansion, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \frac{h^4}{4!}f^{iv}(a) + \dots$$

That is,

$$\begin{aligned}\sin x &= \sin\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\cos\frac{\pi}{4} - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\sin\frac{\pi}{4} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\cos\frac{\pi}{4} + \dots \\ &= \frac{1}{\sqrt{2}}\left(1 + \left(x - \frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} + \dots\right)\end{aligned}$$

(ii) Consider $f(x) = \tan^{-1} x$. Then, we have

$$\begin{aligned}f'(x) &= \frac{1}{1+x^2} \\ f''(x) &= -\frac{2x}{(1+x^2)^2} \\ f'''(x) &= \frac{6x^2-2}{(1+x^2)^3} \\ f^(iv)(x) &= \frac{24(x-x^3)}{(1+x^2)^4} \\ f^v(x) &= \frac{60x^4-240x^2+24}{(1+x^2)^5}\end{aligned}$$

Hence, we have $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -2$

$f^{iv}(x) = 0$ and $f^v(0) = 24$.

Thus, by the Maclaurin's series expansion, we have

$$\tan^{-1} x = x - \frac{x^3}{3} + 24\frac{x^5}{5!} + \dots$$

Lesson - 8

Indeterminate Forms

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Structure

8.1 Learning Objectives

- To learn various type of Indeterminate Forms of limits;
- To understand L' Hospital's Rule and its use in Indeterminate forms;
- To understand the importance of the concepts of Indeterminate forms.

8.2 Introduction

In the previous lessons, we have studied concepts of limits, continuity and differentiabilty. They all are defined on the basic concept of limit of a function. We have studied various techniques to evaluate the limits of functions. Even then, sometimes the problem of finding a limit becomes indeterminate when the established methods fail. In such cases the method of L' Hospital's rule is significantly helpful. We also studied how to find the the limit of a function using numerical methods or graphical evidence. In this chapter, we study an extremely powerful tool known as *L' Hospital Rule*, which is used by many computer program to calculate limits of various type of functions.In this chapter, we shall study an important application of Cauchy's Mean Value Theorem to evaluate a certain type of limits, known as *indeterminate forms*.

From the previous lesson, we have studied that

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}$$

whenever $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and $\lim_{x\to a} g(x)\neq 0$.

In this section, we will study a method to evaluate $\lim_{x\to a}\frac{f(x)}{g(x)}$ even when $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. In this situation, $\lim_{x\to a}\frac{f(x)}{g(x)}$ is said to be of $\frac{0}{0}$ form, which is one of the indeterminate form.

There are several kinds of indeterminate forms such as

$$\frac{0}{0},\frac{\infty}{\infty},0\times\infty,(\infty-\infty),1^{\infty},0^{0}\text{ and }\infty^{0}.$$

But our main focus will be on $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms, which are also known as fundamental forms. All other indeterminate forms can be derived easily with the help of these.

8.3 Indeterminate Form $0/0$

Consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=4,\ \lim_{x\to 0}\frac{\sin x}{x}=1.$$

Here, both the limits are example of indeterminate form $\frac{0}{0}$. The first limit was easily obtained algebraically just by factoring the numerator and denominator and cancelling out the common factor $(x-2)$, but the other limit can't be obtained directly using algebra. Some geometrical approach is needed to evaluate it.

However, there are cases of indeterminate forms, where neither geometric nor algebraic approach work. Therefore, one need to develop some other sophisticated method to deal these type of situations. Following theorem gives a rule to evaluate $\frac{0}{0}$ indeterminate form.

Theorem 8.1 (L' Hospital's Rule for $\frac{0}{0}$ form)**.** *Let f and g be two real-valued functions such that*

(i) $\lim_{x\to a} f(x)=0$ *and* $\lim_{x\to a} g(x)=0$*;*

(ii) *f' and g' both exist and they are continuous at $x=a$ and $g'(a)\neq 0$;*

(iii) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *exists.*

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(a)}{g'(a)}.$$

Proof. Let f and g be two real-valued functions such that f and g both are differentiable and continuous at $x = a$ such that $g'(a) \neq 0$. Therefore, by the continuity of f and g at $x = a$, we have

$$f(a) = \lim_{x \to a} f(x) = 0$$

and

$$g(a) = \lim_{x \to a} g(x) = 0.$$

Now, consider

$$\begin{aligned}
\frac{f'(a)}{g'(a)} &= \frac{\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}}{\lim\limits_{h \to 0} \dfrac{g(a+h) - g(a)}{h}} \\
&= \frac{\lim\limits_{h \to 0} \dfrac{f(a+h)}{h}}{\lim\limits_{h \to 0} \dfrac{g(a+h)}{h}} \\
&= \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} \\
&= \lim_{x \to a} \frac{f(x)}{g(x)}
\end{aligned}$$

□

This result is known as *L' Hospital's Rule*, which converts the $\frac{0}{0}$ indeterminate form into a limit involving the derivatives, which is sometimes easier to calculate.

Remark. The above L' Hospital's Rule is also valid when $x \to \infty$.

In this case, we have

$$\begin{aligned}
\lim_{x \to \infty} \frac{f(x)}{g(x)} &= \lim_{t \to 0} \frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)}, && \text{on substituting } x = \frac{1}{t} \\
&= \lim_{t \to 0} \frac{f'\left(\frac{1}{t}\right) \frac{-1}{t^2}}{g'\left(\frac{1}{t}\right) \frac{-1}{t^2}}, && \text{by using L' Hospital's Rule} \\
&= \lim_{x \to \infty} \frac{f'(x)}{g'(x)}
\end{aligned}$$

Corollary 8.1. Suppose the expression $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exhibits the indeterminate form of the type $\frac{0}{0}$ and the function f' and g' hold all the hypothesis of Theorem 8.1, then the above result can be extended as:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$$

One can generalize the above result as follows:

Theorem 8.2. *Suppose f and g both are real-valued functions such that*

$$\lim_{x\to a}\frac{f^{(n-1)}(x)}{g^{(n-1)}(x)}$$

represents the indeterminate form of the type $\frac{0}{0}$ and the functions $f^{(n)}(x)$ and $g^{(n)}(x)$ hold all the hypothesis of Theorem 8.1, then we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\ldots=\lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}$$

In the following examples, we will demonstrate the L' Hospital's Rule using the following steps:

Working Rule for L' Hospital's Rule

(i) Check whether the $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form $0/0$;

(ii) Differentiate f and g;

(iii) Evaluate the $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$. If the limit exists then it is be equal to $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$.

Example 8.1. Evaluate

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

using L' Hospital's Rule.

Solution. Here, we have $f(x)=x^2-4$ and $g(x)=x-2$ such that

$$\lim_{x\to 2}(x^2-4)=0 \text{ and } \lim_{x\to 2}(x-2)=0$$

Thus, the required limit is an indeterminate form of the type $0/0$. Thus, by using L' Hospital's Rule, we have

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)}=\lim_{x\to 2}\frac{2x}{1}=4.$$

Example 8.2. Using the L' Hospital's Rule, evaluate the following limits

(i) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$;

(ii) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x^3}$;

(iii) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$;

Solution. (i) Here, we have $f(x) = \sin 2x$ and $g(x) = x$ such that

$$\lim_{x\to 0} \sin 2x = 0 = \lim_{x\to 2} x$$

Thus, the required limit is an indeterminate form of the type $0/0$. Therefore, by using L' Hospital's Rule, we have

$$\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \frac{2\cos x}{1} = 2.$$

(ii) Here, we have

$$\lim_{x\to 0} (e^x - 1) = 0 = \lim_{x\to 0} x^3$$

Thus, by using L' Hospital's Rule, we have

$$\lim_{x\to 0} \frac{e^x - 1}{x^3} = \lim_{x\to 0} \frac{e^x}{3x^2} = \infty.$$

(iii) Consider

$$\lim_{x\to 0} (1 - \cos x) = 0 = \lim_{x\to 0} x^2$$

Thus, , we have

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x}.$$

Here, one can observe that $\lim_{x\to 0} \frac{\sin x}{2x}$ is of also a $0/0$ form. Thus, by repeating the L' Hospital's Rule, we get

$$\lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Remark. Consider the following limit:

$$\lim_{x\to 1} \frac{x^2 + x - 2}{5x^3 - 5x^2 + x - 1}$$

Algebraically, one can easily evaluate that

$$\lim_{x\to 1} \frac{x^2 + x - 2}{5x^3 - 5x^2 + x - 1} = \frac{1}{2}$$

by cancelling the common factor $(x - 1)$ from the numerator and denominator.

But, on applying L' Hospital's Rule repeatedly, we get

$$\lim_{x\to 1} \frac{x^2 + x - 2}{5x^3 - 5x^2 + x - 1} = \lim_{x\to 1} \frac{2x + 1}{15x^2 - 10x + 1} = \lim_{x\to 1} \frac{2}{30x - 10} = \lim_{x\to 1} \frac{0}{30x - 10} = 0$$

Now a natural question arises, why the answer obtained by using L' Hospital's Rule is different from the answer obtained from the algebraic method?

For this, consider

$$\lim_{x\to 1} \frac{2x + 1}{15x^2 - 10x + 1}$$

which is obviously not of the form $0/0$, therefore the repeated use of L' Hospital's Rule, without checking the $0/0$ form, is absurd. Hence the required limit is

$$\lim_{x\to 1} \frac{2x + 1}{15x^2 - 10x + 1} = \frac{2 + 1}{15 - 10 + 1} = \frac{1}{2}.$$

In-text Exercise 8.1. *1. Evaluate the following limits:*

$$(i)\ \lim_{x\to 0}\frac{xe^x - \log(1+x)}{x^2};$$

$$(ii)\ \lim_{x\to 0}\frac{1-\cos x^2}{x^2\sin x^2};$$

$$(iii)\ \lim_{x\to 1}\frac{x^2+2x-3}{x^2+3x-4}$$

8.4 Indeterminate Form ∞/∞

Consider the limit

$$\lim_{x\to 0}\frac{\log x^2}{\cot x^2}$$

Here, both, numerator and denominator approaches to ∞ (either $+\infty$ or $-\infty$) as x approaches to 0. Thus, the limit is of the indeterminate form ∞/∞.

Here, we provide L' Hospital's Rule, which helps to evaluate ∞/∞ type of indeterminate form.

Theorem 8.3 (L' Hospital's Rule for ∞/∞ form)**.** *Let f and g be two real-valued functions defined on an closed and bounded interval such that*

(i) *f and g both are differentiable on an open interval about $x = a$ except possibly at $x = a$;*

(ii) $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$*;*

(iii) $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ *exists;*

then, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note. The above theorem also holds in case of a limit as $x\to a^-$, $x\to a^+$, $x\to+\infty$ or as $x\to-\infty$ as well as for $f(x)\to-\infty$ and $g(x)\to-\infty$. If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ is again of the form $\frac{\infty}{\infty}$, then we continue the process by applying L' Hospital's rule to $\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Now, we will illustrate the above theorem with the help of examples.
The working rule is similar to as that for $\frac{0}{0}$ form.

Example 8.3. Evaluate $\lim_{x\to 0}\frac{\log x^2}{\cot x^2}$

Solution. Here

$$\lim_{x\to 0} \log x^2 = \infty \text{ and } \lim_{x\to 0} \cot x^2 = \infty$$

Thus, the required limit is of the form ∞/∞. Therefore, by applying L' Hospital's rule, we have

$$\begin{aligned}\lim_{x\to 0} \frac{\log x^2}{\cot x^2} &= \lim_{x\to 0} \frac{\frac{1}{x^2}2x}{-2x\csc^2 x^2} \\ &= \lim_{x\to 0} \frac{-1}{x^2\csc^2 x^2} \\ &= -\lim_{x\to 0} \frac{\sin x^2}{x^2} \\ &= 1\end{aligned}$$

Example 8.4. Evaluate $\lim\limits_{x\to 0^+} \dfrac{\log x}{\csc x}$.

Solution. Here, we have $f(x) = \log x$ and $g(x) = \csc x$ and the limit of the numerator and denominator are respectively $-\infty$ and $+\infty$. Thus, we have an indeterminate form of the type ∞/∞. Therefore, by applying L' Hospital's Rule, we have

$$\lim_{x\to 0^+} \frac{\log x}{\csc x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x} \tag{8.1}$$

The limit in Equation 8.1 is again of the indeterminate form of type ∞/∞. However, repeated use of L' Hospital's Rule for this gives us the power of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator which again generates new indeterminate form. Therefore, from this point, one must use another methods (algebraic or geometrical one) to solve this. Thus, after simplification, Equation 8.1 can be written as

$$\lim_{x\to 0^+} \left(-\frac{\sin x}{x}\tan x\right) = -\lim_{x\to 0^+} \frac{\sin x}{x}. \lim_{x\to 0^+} \tan x = -1.0 = 0$$

Thus, we have

$$\lim_{x\to 0^+} \frac{\log x}{\csc x} = 0.$$

Example 8.5. Evaluate $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x}$, for $n \in \mathbb{N}$.

Solution. Consider $f(x) = x^n$ and $g(x) = e^x$. Here

$$\lim_{x\to\infty} \frac{x^n}{e^x} \text{ is of the form } \frac{0}{0}$$

On repeated use of L' Hospital's Rule, we have

$$\begin{aligned}\lim_{x\to\infty}\frac{x^n}{e^x} &= \lim_{x\to\infty}\frac{nx^{n-1}}{e^x}\\ &= \lim_{x\to\infty}\frac{n(n-1)x^{n-1}}{e^x}\\ &\ldots\ldots\ldots\ldots\ldots\\ &= \lim_{x\to\infty}\frac{n(n-1)(n-2)\ldots 3.2.1}{e^x}\\ &= \lim_{x\to\infty}\frac{n!}{e^x} = 0.\end{aligned}$$

In-text Exercise 8.2. *1. Evaluate the following limits:*

(a) $\lim_{x\to\infty}\frac{e^x}{x^3}$;

(b) $\lim_{x\to 0}\log_{\sin x}\sin 2x$;

(c) $\lim_{x\to 1}\frac{\log\tan x}{\log\tan 2x}$

8.5 Indeterminate form $0.\infty$ and $\infty - \infty$

Consider the limit

$$\lim_{x\to 0^+} x\ln x$$

Here, it is of the form $0.\infty$, as $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} x = -\infty$. These two limits exert conflicting influences on the product.

Indeterminate form of the types $0.\infty$ can be manipulated by rewriting the product as of the form either $0/0$ or ∞/∞ and then can be solved by using L' Hospital's Rule.

Let $f(x)$ and $g(x)$ be two real valued functions such that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and } \lim_{x\to a} g(x) = \infty$$

then

$$\lim_{x\to a} f(x).g(x) = 0.\infty$$

form. Then, we can write

$$\lim_{x\to a} f(x).g(x) = \lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}} \text{ or } \lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$$

which are $\frac{0}{0}$ and $\frac{\infty}{\infty}$ forms respectively and can be evaluated by L' Hospital's Rule.

Example 8.6. Evaluate

$$\lim_{x\to\pi/2}(1-\sin x)\tan x$$

Solution. Here, we have $f(x) = 1 - \sin x$ and $g(x) = \tan x$, as $x \to \pi/2$, the given project leads to $0.\infty$ form. Hence, we have

$$\begin{aligned}\lim_{x\to\pi/2}(1-\sin x)(\tan x) &= \lim_{x\to\pi/2}\frac{1-\sin x}{\tan x} \qquad \left(\frac{0}{0}\right)\\ &= \lim_{x\to\pi/2}\frac{-\cos x}{\csc^2 x} \qquad (\text{ by L' Hospital's Rule})\\ &= 0.\end{aligned}$$

Example 8.7. Evaluate

$$\lim_{x\to 0^+} x\ln x$$

Solution. Here, the required limit is of the form $0.\infty$. Thus, we can evaluate the limit by two different ways:

$$\lim_{x\to 0^+}\frac{\ln x}{1/x} \qquad \text{or} \qquad \lim_{x\to 0^+}\frac{x}{1/\ln x}$$

Here, first one is indeterminate form of type ∞/∞ and the second one is of $0/0$.

We will apply the first form to evaluate the same. Therefore

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+}\frac{\ln x}{1/x}\\ &= \lim_{x\to 0^+}\frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+}(-x)\\ &= 0.\end{aligned}$$

A limit which leads to one of the following forms

$$(+\infty)-(+\infty), (-\infty)-(-\infty)$$
$$(+\infty)+(-\infty), (-\infty)+(+\infty)$$

is known as indeterminate form of the type $\infty - \infty$. Such limits are indeterminate form because of the fact that the two terms exert conflicting influences on the expression, in which one leads the term to extreme positive direction and the other one leads the same in the extreme negative direction.

Let $f(x)$ and $g(x)$ be two real valued functions such that

$$\lim_{x\to a} f(x) = \infty \qquad \text{and } \lim_{x\to a} g(x) = \infty$$

then

$$\lim_{x\to a}(f(x) - g(x))$$

is of the form $\infty - \infty$. Then, we write

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a}\frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x).g(x)}}$$

which is of the form $\frac{0}{0}$ form and can be evaluated accordingly.

Example 8.8. Evaluate

$$\lim_{x\to 0}\left(\cot^2 x - \frac{1}{x^2}\right)$$

Solution. Consider $f(x) = \cot^2 x - \frac{1}{x^2}$, as $x \to 0$, then $f(x)$ approaches to $\infty - \infty$ form. Then, we have

$$\begin{aligned}
\lim_{x\to 0}\left(\cot^2 x - \frac{1}{x^2}\right) &= \lim_{x\to 0}\frac{x^2\cos^2 x - \sin^2 x}{x^2\sin^2 x} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{x^2\cos^2 x - \sin^2 x}{x^4}.\frac{x^2}{\sin^2 x} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{x^2\cos^2 x - \sin^2 x}{x^4} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{x^2(1+\cos 2x) - (1-\cos 2x)}{2x^4} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{x^2 + (x^2+1)\cos 2x - 1}{2x^4} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{2x + 2x\cos 2x - 2(x^2+1)\sin 2x}{8x^3} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{2 + 2\cos 2x - 8x\sin 2x - 4(x^2+1)\cos 2x}{24x^2} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{-4\sin 2x - 8\sin 2x - 24x\cos 2x + 8(x^2+1)\sin 2x}{48x} && \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{x\to 0}\frac{-48\cos 2x + 64\sin 2x + 16(x^2+1)\cos 2x}{48}\\
&= \frac{-48+16}{48} = -\frac{2}{3}
\end{aligned}$$

Example 8.9. Evaluate

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$$

Solution. Here, both the terms tends to ∞, therefore the required limit is of the form $\infty - \infty$, which can be rewritten as follow:

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+}\frac{\sin x - x}{x\sin x}$$

The new modified limit is an indeterminate form of the type $0/0$. Thus, after repeated use of L' Hospital's Rule, we will get

$$\begin{aligned}
\lim_{x\to 0^+}\frac{\sin x - x}{x\sin x} &= \lim_{x\to 0^+}\frac{\cos x - 1}{\sin x + x\cos x}\\
&= \lim_{x\to 0^+}\frac{-\sin x}{\cos x + \cos x - x\sin x}\\
&= \frac{0}{2} = 0
\end{aligned}$$

In-text Exercise 8.3. *1. Evaluate the following limits*

(a) $\lim_{x\to\frac{\pi}{2}}\left(\sec x - \frac{1}{1-\sin x}\right)$

(b) $\lim_{x\to 1}\left(\frac{2}{x^2-1} - \frac{1}{x-1}\right)$*;*

(c) $\lim_{x\to\frac{\pi}{2}}\left(x\tan x - \frac{\pi}{2}\sec x\right)$

8.6 Indeterminate Forms $0^0, \infty^0, 1^\infty$

Consider

$$\lim_{x\to 0}(1+x)^{1/x}.$$

The value of the above mentioned limit is e but it is an indeterminate form of the type 1^∞. It is because of the fact that as x approaches to 0, the expression $1+x$ tends to 1 but $1/x$ approaches to ∞, both exert two conflicting influences.

Thus, the limit

$$\lim_{x\to a} f(x)^{g(x)}$$

is of the form 0^0 or ∞^0 or 1^∞.

when $\lim_{x\to a} f(x) = 0, 1$ or ∞ and $\lim_{x\to a} g(x) = 0, \infty$ or 0.

To evaluate such limits, we write

$$y = f(x)^{g(x)}$$

$$\Rightarrow \quad \log y = \log f(x)^{g(x)}$$

$$\text{Therefore } \lim_{x\to a}\log y = \lim_{x\to a} g(x)\log f(x)$$

Now, the limit in the RHS is of the form $0.\infty$ when $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a}\log f(x) = \infty$. This can be evaluated by using L' Hospital's Rule.

Let us consider

$$\lim_{x\to a} g(x)\log f(x) = l$$

$$\lim_{x\to a}\log y = l$$

$$\log\lim_{x\to a} y = l$$

$$\lim_{x\to a} y = e^l$$

Hence, we have

$$\lim_{x\to a} f(x)^{g(x)} = e^l$$

Example 8.10. Evaluate

$$\lim_{x\to 0}(1+\sin x)^{1/x}$$

Solution. The required limit is of 1^∞ form. Let

$$y = (1 + \sin x)^{1/x}$$

Taking logarithm on both sides, we get

$$\begin{aligned}\log y &= \log\,(1 + \sin x)^{1/x}\\ &= \frac{1}{x}\log(1 + \sin x)\\ &= \frac{\log(1 + \sin x)}{x}\end{aligned}$$

Thus, we have

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log(1 + \sin x)}{x}$$

which is of the form $0/0$. Therefore, by applying L' Hospital's Rule, we get

$$\begin{aligned}\lim_{x\to 0} \log y &= \lim_{x\to 0} \frac{\log(1 + \sin x)}{x}\\ &= \lim_{x\to 0} \frac{\cos x/(1 + \sin x)}{1}\\ &= 1\end{aligned}$$

Hence, we get $\log y$ tends to 1 as $x \to 0$ and by using continuity of exponential function, we have

$$e^{\log y} \to e^1 \text{ as } x \to 0$$

Therefore,

$$\lim_{x\to 0}(1 + \sin x)^{1/x} = e$$

Example 8.11. Evaluate

$$\lim_{x\to \pi/2} (\sin x)^{\tan x}$$

Solution. Let

$$y = (\sin x)^{\tan x}$$

Thus, taking logarithm on both side, we get

$$\begin{aligned}\log y &= \log\,(\sin x)^{\tan x}\\ &= \tan x \log(\sin x)\\ \lim_{x\to \pi/2} \log y &= \lim_{x\to \pi/2} \tan x \log(\sin x)\\ &= \lim_{x\to \pi/2} \frac{\log \sin x}{\cot x}\\ &= \lim_{x\to \pi/2} \frac{\cos x}{\sin x(-\csc^2 x)}\\ &= 0.\end{aligned}$$

Thus, we have

$$\lim_{x\to\pi/2} \log y = 0 \Rightarrow \lim_{x\to\pi/2} y = e^0 = 1$$

Hence, we get

$$\lim_{x\to\pi/2} (\sin x)^{\tan x} = 1$$

In-text Exercise 8.4. *1. Evaluate*

(a) $\lim_{x\to 0} x^x$;

(b) $\lim_{x\to 0} \left(\frac{2x+1}{x+1}\right)^{1/x}$.

8.7 Summary

In this chapter, we have covered the following:

1. **L' Hospital's Rule for $\frac{0}{0}$ form**

   Let f and g be two real-valued functions such that

   (i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$;

   (ii) f' and g' both exist and continuous at $x = a$ and $g'(a) \neq 0$;

   (iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

   Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

2. **Working Rule for L' Hospital's Rule**

   (i) check whether the limit of $f(x)/g(x)$ is of the form $0/0$;

   (ii) differentiate f and g;

   (iii) evaluate the limit of $f'(x)/g'(x)$. If the limit exists then it should be equal to the limit of $f(x)/g(x)$.

8.8 Self-Assessment Exercises

1. Evaluate the following limits in indeterminate forms:

   (a) $\lim_{x\to 0} \frac{e^x - 1}{\sin x}$;

   (b) $\lim_{x\to 0} \frac{\tan x}{x}$;

(c) $\lim_{x \to 0} \dfrac{xe^x}{1 - e^x}$;

(d) $\lim_{x \to 0} \dfrac{\sin x}{x^2}$;

(e) $\lim_{x \to 0} \dfrac{\csc x}{\log x}$;

(f) $\lim_{x \to 1} \dfrac{\log(1 - x)}{\cot \pi x}$;

(g) $\lim_{x \to 0^+} \dfrac{1 - \log x}{e^{1/x}}$;

(h) $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$;

8.9 Solutions to In-text Exercises

Exercise 8.1

1. Consider $f(x) = xe^x - \log(1 + x)$ and $g(x) = x^2$. One can easily observe that as x tends to zero, f and g both tends to zero. Thus, the given limit is a $0/0$ indeterminate form. Thus, by applying L' Hospital's Rule, we get

$$\begin{aligned} \lim_{x \to 0} \frac{f(x)}{g(x)} &= \lim_{x \to 0} \frac{f'(x)}{g'(x)} \\ &= \lim_{x \to 0} \frac{e^x + xe^x - \frac{1}{1+x}}{2x} \\ &= \lim_{x \to 0} \frac{e^x(2 + x) + \frac{1}{(1+x)^2}}{2} \\ &= \frac{3}{2}. \end{aligned}$$

2. Consider $f(x) = 1 - \cos x^2$ and $g(x) = x^2 \sin x^2$. As x tends to zero, f and g both tends to zero. Thus, we have $0/0$ indeterminate form. Hence

$$\begin{aligned} \lim_{x \to 0} \frac{f(x)}{g(x)} &= \lim_{x \to 0} \frac{f'(x)}{g'(x)} \\ &= \lim_{x \to 0} \frac{2x \sin x^2}{2x^3 \cos x^2 + 2x \sin x^2} \qquad \left(\frac{0}{0} \text{ Indeterminate form}\right) \\ &= \lim_{x \to 0} \frac{4x^2 \cos x^2 + 2 \sin x^2}{10x^2 \cos x^2 + 2 \sin x^2 - 4x^2 \sin x^2} \qquad \left(\frac{0}{0} \text{ Indeterminate form}\right) \\ &= \lim_{x \to 0} \frac{12x \cos x^2 - 8x^3 \sin x^2}{24x \cos x^2 - 8x^5 \cos x^2 - 36x^3 \sin x^2} \qquad \left(\frac{0}{0} \text{ Indeterminate form}\right) \\ &= \lim_{x \to 0} \frac{12 \cos x^2 - 16x^4 \cos x^2 - 48x^2 \sin x^2}{24 \cos x^2 - 112x^4 \cos x^2 - 156x^2 \sin x^2 + 16x^6 \sin x^2} \\ &= \frac{1}{2}. \end{aligned}$$

3. We have $f(x) = x^2 + 2x - 3$ and $g(x) = x^2 + 3x - 4$. Here, both f and g approaches to zero as x tends to 1. Thus, the given limit is of $0/0$ indeterminate form. Therefore, by using L' Hospital's Rule, we have

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\frac{f'(x)}{g'(x)}\\ &= \lim_{x\to 0}\frac{2x+2}{2x+3}\\ &= \frac{4}{5}.\end{aligned}$$

Exercise 8.2

1. Here, we have $f(x) = e^x$ and $g(x) = x^3$. As x tends to ∞, the given limit becomes ∞/∞ form. Thus, by applying L' Hospital's Rule, we get

$$\begin{aligned}\lim_{x\to\infty}\frac{e^x}{x^3} &= \lim_{x\to\infty}\frac{e^x}{3x^2} \quad \left(\frac{\infty}{\infty}\text{ Indetermiate Form}\right)\\ &= \lim_{x\to\infty}\frac{e^x}{6x} \quad \left(\frac{\infty}{\infty}\text{ Indetermiate Form}\right)\\ &= \lim_{x\to\infty}\frac{e^x}{6}\\ &= \infty\end{aligned}$$

2. We have $f(x) = \log_{\sin x}\sin 2x = \frac{\log\sin x}{\log\sin 2x}$. Now, consider

$$\lim_{x\to 0}\frac{\log\sin x}{\log\sin 2x}$$

which is of ∞/∞ form. Thus, by applying L' Hospital's Rule, we get

$$\begin{aligned}\lim_{x\to 0}\frac{\log\sin x}{\log\sin 2x} &= \lim_{x\to 0}\frac{\cot x}{2\cot 2x} \quad \left(\frac{\infty}{\infty}\text{ Indetermiate Form}\right)\\ &= \lim_{x\to 0}\frac{-\csc^2 x}{-4\csc^2 2x} \quad \left(\frac{\infty}{\infty}\text{ Indetermiate Form}\right)\\ &= \lim_{x\to 0}\cos^2 x\\ &= 1.\end{aligned}$$

3. We have

$$\begin{aligned}\lim_{x\to 0}\frac{\log\tan x}{\log\tan 2x} &= \lim_{x\to 0}\frac{\frac{1}{\tan 2x}2\sec^2 2x}{\frac{1}{\tan x}\sec^2 x} \quad \left(\frac{\infty}{\infty}\text{ Indetermiate Form}\right)\\ &= \lim_{x\to 0}\frac{2\sin 2x}{\sin 4x} \quad \left(\frac{\infty}{\infty}\text{ Indetermiate Form}\right)\\ &= \lim_{x\to 0}\frac{4\cos 2x}{4\cos 4x}\\ &= 1.\end{aligned}$$

Exercise 8.3

1. We have

$$\lim_{x\to\frac{\pi}{2}}\left(\sec x - \frac{1}{1-\sin x}\right)$$

As $x \to \pi/2$, the required limit is of $\infty - \infty$ form. After simplification, we have

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}}\left(\sec x - \frac{1}{1-\sin x}\right) &= \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x\cos x}{\cos x(1-\sin x)} \quad (\infty-\infty \text{ Indetermiate Form})\\
&= \lim_{x\to\frac{\pi}{2}}\frac{-\cos x+\sin x}{\cos^2 x-(1-\sin x)\sin x} \quad (\infty-\infty \text{ Indetermiate Form})\\
&= \lim_{x\to\frac{\pi}{2}}\frac{\cos x+\sin x}{-\cos x(1-\sin x)+3\cos x\sin x} \ (\infty-\infty \text{ Indetermiate Form})\\
&= \lim_{x\to\frac{\pi}{2}}\frac{\cos x-\sin x}{4\cos^2 x+(1+\sin x)\sin x-3\sin x^2}\\
&= -\infty
\end{aligned}$$

2. Consider

$$\begin{aligned}
\lim_{x\to 1}\left(\frac{1}{x^2-1}-\frac{1}{x-1}\right) &= \lim_{x\to 1}\left(\frac{2-x-1}{x^2-1}\right) \ (\infty-\infty \text{ Indetermiate Form})\\
&= \lim_{x\to 1}\left(\frac{-1}{x+1}\right)\\
&= -\frac{1}{2}
\end{aligned}$$

3. We have

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}}\left(x\tan x-\frac{\pi}{2}\sec x\right) &= \lim_{x\to\frac{\pi}{2}}\left(\frac{2x\sin x-\pi}{2\cos x}\right) \ (\infty-\infty \text{ Indetermiate Form})\\
&= \lim_{x\to\frac{\pi}{2}}\frac{2x\cos x+2\sin x}{-2\sin x} \ (\infty-\infty \text{ Indetermiate Form})\\
&= \frac{\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)+2\sin\left(\frac{\pi}{2}\right)}{-2\sin\left(\frac{\pi}{2}\right)}\\
&= -1.
\end{aligned}$$

Exercise 8.4

1. Consider

$$y = x^x$$

Taking $\log$ on both sides, we have

$$\begin{aligned}
\log y &= \log x^x\\
\log y &= x\log x
\end{aligned}$$

Thus, we have

$$\lim_{x \to 0} x \log x = 0$$

Therefore, $\lim_{x \to 0} \log y = 0$ and

$$\lim_{x \to 0} y = e^0 = 1.$$

2. We have

$$y = \left(\frac{2x+1}{x+1}\right)^{1/x}$$

Taking $\log$ on both sides, we have

$$\begin{aligned}\log y &= \frac{1}{x}\left(\log\left(\frac{2x+1}{x+1}\right)\right) \\ &= \frac{\log(2x+1)}{x} - \frac{\log(x+1)}{x}\end{aligned}$$

Thus, we have

$$\begin{aligned}\lim_{x \to 0}\left(\frac{\log(2x+1)}{x} - \frac{\log(x+1)}{x}\right) &= \lim_{x \to 0}\left(\frac{\frac{2}{2x+1}}{1} - \frac{\frac{1}{x+1}}{1}\right) \quad (\infty - \infty \text{ Indetermiate Form}) \\ &= \lim_{x \to 0}\frac{1}{(x+1)(2x+1)} \\ &= 1.\end{aligned}$$

Hence, we have

$$\lim_{x \to 0} y = e^1 = e$$

Suggested Readings

- Anton, Howard, Bivens, Irl, & Davis, Stephen (2013). Calculus (10th ed.). Wiley India Pvt. Ltd. New Delhi. International Student Version. Indian Reprint 2016.
- Arora, S. C., Kumar, Ramesh. A text book of Calculus. Delhi. 2016.
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Lesson - 9

Concavity And Asymptotes

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Structure

9.1 Learning Objectives

- To understand the concept of increasing and decreasing functions.
- To understand concave up and concave down curves (functions).
- To learn technique to examine concavity
- To learn about a point of inflection and method to find it.
- To learn about asymptotes of a curve.

- To study the techniques to obtain different type of asymptotes .

9.2 Introduction

We have learned that the graph of a function f shows where the slope is increasing or decreasing depending on the sign of the derivative of f. We can see from definition that when f is defined on an interval, x_1 and x_2 signify the points in the interval, if $f(x_1) < f(x_2)$ then f is increasing, if $f(x_1) > f(x_2)$ then f is decreasing and f is constant on the interval if $f(x_1) = f(x_2)$. But this definition does not explain the direction of curvature. Now we will study about the curvature of a graph.We are interested in the graph of f or the shape of the graph of a f. In this section, we will study the graph and develop the mathematical tools for graph using the derivatives of the function. In this lesson we will discuss all those tangents of a curve who touch the graph at infinity. These are tangents of special nature, known as asymptotes. There are four main types of asymptotes. This section also explain how to find these asymptotes for a given curve.

9.3 Concavity

Definition 9.1. If f is differentiable on an open interval I then f is said to be concave up on I, if f' is increasing on I, and f is said to be concave down on I if f' is decreasing on I.

Remark. 1. If tangent line of function f at a point $x = c$ lies below the graph then it is concave up at $x = c$.

2. If tangent line of function f at a point $x = c$ lies above the graph then it is concave down at a point $x = c$.

3. If the tangent line of f has rising slopes on an open interval then f is concave up; otherwise f is concave down.

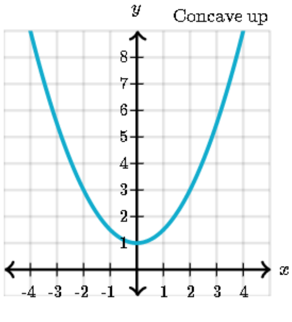

Figure 9.1: Concave Up function

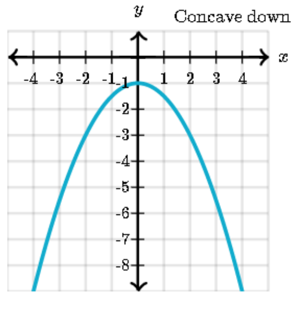

Figure 9.2: Concave Down function

Figure 9.1 shows the graph of concave up function and Figure 9.2 shows the graph of Concave down function.

Theorem 9.1. *Let f be twice differentiable on an open interval I.*

a) If $f''(x) > 0$ for every value of x in the open interval I, then f is concave up on I.

b) If $f''(x) < 0$ for every value of x in the open interval I, then f is concave down on I.

Example 9.1. If $f(x) = x^2 - 4x + 3$, then find the interval where it is concave up and concave down.

Solution. We have, $f(x) = x^2 - 4x + 3$

$$\begin{aligned} \implies \quad & f'(x) = 2x - 4 \\ \text{and} \quad & f''(x) = 2 > 0 \text{ for all x} \end{aligned}$$

Hence $f(x)$ is concave up on $(-\infty, \infty)$

Example 9.2. If $f(x) = x^3$, then find the interval where it is concave up and concave down.

Solution. We have $f(x) = x^3$

$$\begin{aligned} &\implies f'(x) = 3x^2 \\ &\implies f''(x) = 6x \end{aligned}$$

Therefore, $\quad f''(x) = 0$

$\implies \quad 6x = 0$

$\implies \quad x = 0$

Case I : $x \in (-\infty, 0)$

$$f''(x) = 6x < 0$$

$\implies$ f is concave down on the interval $(-\infty, 0)$

Case II : $x \in (0, \infty)$

$$f''(x) = 6x > 0$$

$\implies$ f is concave up on the interval $(0, \infty)$

9.4 Point of Inflection

We have seen in **Example 9.2** that the concavity changes at the point $x = 0$. That is on right hand side of $x = 0$ the graph $f(x) = x^3$ is concave up and on left hand side it is concave down. Points where a curve changes its concavity are the points of inflection.

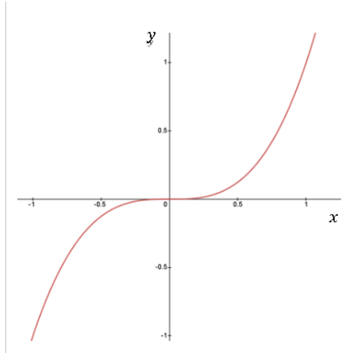

Figure 9.3: $f(x) = x^3$

Definition 9.2. If f is continuous on an open interval containing a point x_0, and if f changes the direction of concavity at the point $(x_0, f(x_0))$, then we say that f has a point of inflection at x_0. The point $(x_0, f(x_0))$ on the graph of f is the point of inflection point of f. It can be noted that the any tangent line crosses the curve at a point of inflection.

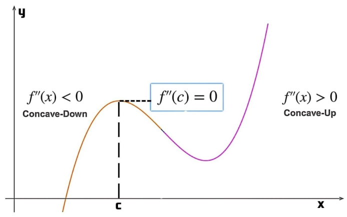

Figure 9.4:

In Figure 9.4 we can see that $x = c$ is an inflection point.

Example 9.3. Find inflection point for the function

$$f(x) = x^3 - 3x^2 + 2$$

Solution. We have, $f(x) = x^3 - 3x^2 + 2$

$$\implies f'(x) = 3x^2 - 6x,$$
$$f''(x) = 6x - 6 = 6(x-1)$$

Therefore,

$$f''(x) = 0 \quad \implies \quad 6(x-1) = 0$$
$$x - 1 = 0,$$
$$\implies x = 1$$

Case I : If $x < 1$. Then

$$f''(x_) = 6(x-1) < 0$$

$\implies$ f is concave down in $(-\infty, 1)$.
Case II : If $x > 1$. Then

$$f''(x_) = 6(x-1) > 0$$

$\implies$ f is concave up in $(1, \infty)$.
Since,f changes its behaviour from concave downward to concave upward ar $x = 1$. Therefore $x = 1$ is an inflection point.

Example 9.4. Find inflection point for the function

$$f(x) = 2x + 3\sin x$$

Solution. We have, $f(x) = 2x + 3\sin x$

$$\implies f'(x) = 2 + 3\cos x,$$
$$f''(x) = -3\sin x$$

Therefore,

$$f''(x) = 0 \implies \text{-3}\sin x = 0,$$
$$\sin x = 0$$
$$\implies x = 0, \pi, 2\pi$$

Case I : $0 < x < \pi$

$$f''(x) = -3\sin x < 0$$

$\implies$ f is concave down in $(0, \pi)$.
Case II : $\pi < x < 2\pi$

$$f''(x) = -3\sin x > 0$$

$\implies$ f is concave up in $(\pi, 2\pi)$.
Hence, $x = \pi$ is an inflection point

Example 9.5. Find inflection point for the function.

$$f(x) = x^4 + 1$$

Solution. We have, $f(x) = x^4 + 1$

$$f'(x) = 4x^3$$
$$\implies f''(x) = 12x^2$$

Therefore,

$$f''(x) = 0 \implies 12\text{x}^2 = 0$$
$$\implies x = 0$$

Case I : $x < 0$. Then

$$f''(x_) = 12x^2 > 0$$

$\implies f$ is concave up in $(-\infty, 0)$.
Case II : $x > 0$

$$f''(x_) = 12x^2 > 0$$

$\implies f$ is concave up in $(0, \infty)$.
Since, f does not change its behaviour at $x = 0$, therefore $x = 0$ is not a point of inflection
Hence,There is no inflection point.

In-text Exercises 9.1

Search the open intervals where f is concave up and concave down. Additionally, identify each x-coordinates of inflection points.

1. $f(x) = x^2 - 3x + 8$
2. $f(x) = (2x + 1)^3$
3. $f(x) = 3x^4 - 4x^3$
4. $f(x) = \frac{x-2}{(x^2-x+1)^2}$

9.5 Asymptotes

A straight line that continuously approaches a specific curve but does not intersect it at any finite point is called an Asymptote. That is, an Asymptote, is a straight line which touches the curve at infinity.

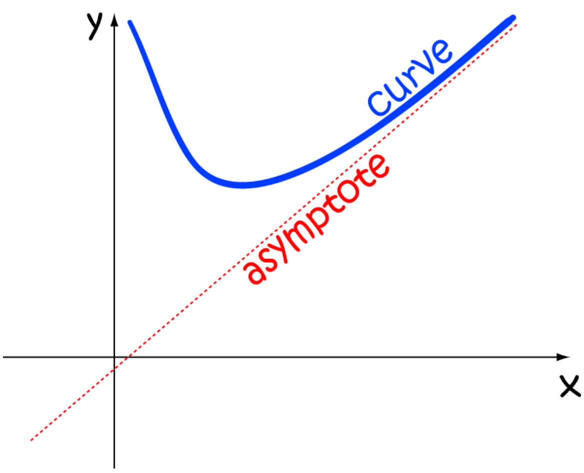

Figure 9.5:

9.5.1 The formal definition of an Asymptote

A straight line is called an asymptote of the given curve, if the distance between the curve and the line approches to zero as x and y coordinates tends to infinity.

9.6 Types of Asymptotes

9.6.1 Type-I: Asymptotes Parallel to the x-axis or Horizontal Asymptotes

Working Rule: A horizontal asymptotes or the asymptotes parallel to x-axis are determined equating zero to the coefficient of highest power of x, provided this coefficient is not a constant.

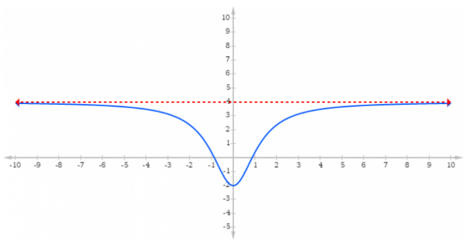

Figure 9.6: Horizontal Asymptote

Example 9.6. Find a asymptote, parallel to the axis of x, of the curve $x^2y+2xy+y+1 = 0$.

Solution. y=0 is the asymptote parallel to the x-axis as the coefficient of the highest power of x in y.

Example 9.7. Find a asymptote, parallel to the axis of x, of the curve $3y^4 + x^2y^2 + 2xy^2 - 4x^2 + 4x + 2y + 1 = 0$.

Solution. $(y^2 - 4) = 0 \implies y^2 = 4 \implies y = \pm 2$, as the coefficient of the highest power of x in $y^2 - 4$
Hence $y = 2$ and $y = -2$ are the two asymptotes parallel to the x-axis.

9.6.2 Type-II : Asymptotes parallel to the y-axis or Vertical Asymptotes

Working Rule: A vertical asymptotes or the asymptotes parallel to y-axis are determined equating zero to the coefficient of highest power of y, provided this coefficient is not a constant.

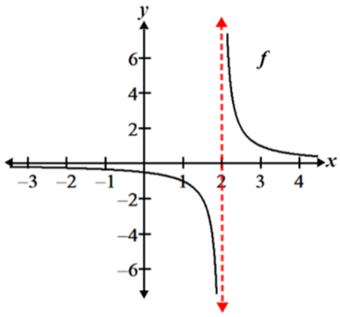

Figure 9.7: Vertical Asymptote

Example 9.8. Find a asymptote, parallel to the axis of y, of the curve
$x^3 + x^2y^2 - a^2(x^2 + y^2) = 0$.

Solution. Since the coefficient of highest power of y is $x^2 - a^2$, therefore we take $x^2 - a^2 = 0 \implies x^2 = a^2 \implies x = \pm a$
Hence, $x = a$ and $x = -a$ are the two asymptote parallel to the y-axis.

Example 9.9. Find a asymptote, parallel to the axis of y, of the curve $x^2y + a^2xy^2 + 2xy + 2x + y + 1 = 0$.

Solution. Since the coefficient of highest power of y is a^2x, therefore we have $a^2x = 0 \implies x = 0$
Hence, $x = 0$ is the two asymptote parallel to the y-axis.

9.6.3 Type-III : Oblique Asymptote

When x moves toward ∞ or$-\infty$, the curves moves towards a line $y = mx + c$, then this line $y = mx + c$ is called Oblique Asymptote.

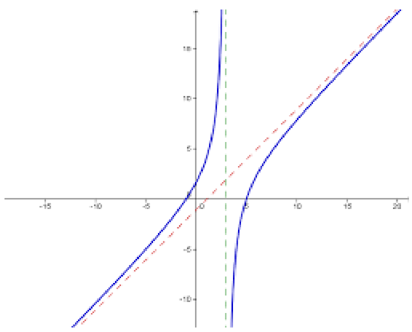

Figure 9.8: Oblique Asymptote

Working rule for finding Oblique Asymptotes
Method I:
Step 1 : Put $mx + c$ at the place of y in the equation.
Step 2 : The coefficient of two highest power of x should be equal to 0.

Step 3 : Determine m and c from the equations getting in **Step 2**, say (m_1, c_1) , (m_2, c_2) , (m_3, c_3) , ...
Step 4 : Now we get the asymptotes
$y = m_1x + c_1, y = m_2x + c_2, y = m_3x + c_3$, ...
Method II:
Step 1 : Put $y = m$ and $x = 1$ in the highest degree term and getting a polynomial in m say $\phi_n(m)$.(n represent the degree of polynomial).
Step 2 : Put $\phi_n(m) = 0$ and find the values of m, say m_1, m_2, m_3,...
Step 3 : Put $y = m$ and $x = 1$ in the second highest degree term and getting a polynomial in m say $\phi_{n-1}(m)$.(n represent the degree of polynomial)
Step 4 : Find the values of c from the formula

$$c = -\frac{\phi_{n-1}(m)}{\phi_n(m)}$$

say c_1, c_2, c_3,...corresponding to m_1, m_2, m_3,...
Step 5 : Now we get asymptotes

$$\begin{gathered} y = m_1x + c_1, \\ y = m_2x + c_2, \\ y = m_3x + c_3 \\ ... \\ ... \end{gathered}$$

Example 9.10. :Find the asymptotes of

$$x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 = 2$$

Solution. Putting $x = 1$ and $y = m$, in the highest degree i.e. 3rd and second highest degree i.e. 2nd degree terms in given equation, we get
$\phi_3(m) = 1 + 2m - m^2 - 2m^3$; $\quad \phi_2(m) = m - m^2$
Then the slopes of asymptotes of given equation will be given by
$\phi_3(m) = 0 \quad$ i.e. $1 + 2m - m^2 - 2m^3 = 0$
or $(1 - m)(1 + m)(1 + 2m) = 0$ whence m=1,-1,-1/2.
Now $\phi_3'(m) = 2 - 2m - 6m^2 = 2(1 - m - 3m^2)$
Then $c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{m^2-m}{2(1-m-3m^2)} = 0, -1, 1/2$
for m=1,-1,-1/2 respectively. Hence the required asymptotes are

$$y = x, \quad y + x + 1 = 0, \quad and \quad 2y + x = 1$$

9.6.4 Type-IV : Parallel Asymptotes

When we get two real equal values of m from the equation $\phi_n(m) = 0$, say $m_1 = m_2 = m$ and if $\phi_n'(m) = 0$ and $\phi_{n-1}(m) = 0$, then the curve will have two parallel asymptotes.

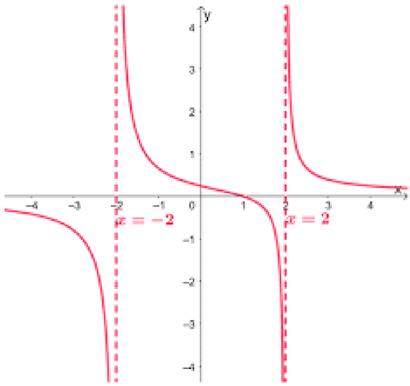

Figure 9.9: Parallel Asymptote

Working rule for finding two prallel asymptotes :
Step 1 : Put $y = mx + c$ at the place of y in the equation.
Step 2 : The coefficient of highest power of x should be equal to 0 and get the equation in m and say $\phi_n(m) = 0$.
Step 3 : Solve the equation $\phi_n(m) = 0$ and find the values of m.
Step 4 : If any value of m from **Step 3** makes the coefficient of second highest degree of x (i.e. x^{n-1}) is zero, then the value of c should be calculated using the equation created by substituting the coefficients of x^{n-1} equal to zero.
Then, $y = mx + c$ is an asymptote

Example 9.11. Find the asymptotes of $y^3 + x^2y + 2xy^2 - y + 5 = 0$.

Solution. Putting $y = mx + c$ in the above equation, we have
$(mx + c)^3 + x^2(mx + c) + 2x(mx + c)^2 - (mx + c) + 5 = 0$
or $x^3(m^3 + m + 2m^2) + x(3m^2c + c + 4mc) + x(3mc^2 + 2c^2 - m) + ... = 0$.
Equating to zero, the coefficients of x^3 and x^2, we get
$m^3 + 2m^2 + m = 0$ and $(3m^2 + 4m + 1)c = 0$.
$m^3 + 2m^2 + m = 0 \implies m = 0, -1, -1$ If $m = 0$, the equation with second higest degree of x gives $c = 0$
If $m = -1$, the second equation reduces to $0.c = 0$. In such case, equating the coefficient of the next highest power of x to zero (or by finding $\phi''(m)$ $\phi_{n-2}(m)$), we get $(3m + 2)c - m = 0$. From this equation we get $c = 1$ when $m = -1$.
Now we get the asymptotes

$$y = 0, \;\; y = -x + 1, \;\; y = -x - 1$$

Example 9.12. Find the asymptotes of the cubic curve
$y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x - 1 = 0$.

Solution. Putting $y = mx + c$ in the equation, we obtain
$(mx + c)^3 - 5x(mx + c)^2 + 8x^2(mx + c) - 4x^3 - 3(mx + c)^2 + 9x(mx + c) - 6x^2 + 2(mx + c) - 2x = 1$

or $x^3(m^3-5m^2+8m-4)+x^2(3m^2c-10mc+8c-3m^2+9m-6)+.....=0$
Equating to zero, the coefficients of x^3 and x^2, we get
$m^3-5m^2+8m-4=0$, and $(3m^2-10m+8)c-3m^2+9m-6=0.$
The first equation gives $m=1,2,2$
If $m=1$, then the second equation gives $c=0$. Hence, the corresponding asymptote is $y=x$.
If $m=2$, the second equation reduces to the identity 0. $c+0=0$. Therefore, to find c, equate the next lower coefficient (of x) to zero in equation (1). This gives
$3mc^2-5c^2-6mc+9c+2m-2=0.$
Putting $m=2$ in this, we get
$c^2-3c+2=0 \ \ or \ \ c=1 \ \ or \ \ 2.$
So, for $m=2$, there are two parallel asymptotes $y=2x+1 \ \ and \ \ y=2x+2$
Hence the asymptotes are $y=x, \ \ y=2x+1 \ \ and \ \ y-2x+2$

In-text Exercises 9.2

Find the asymptotes of the following curves :

1) $x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0$
2) $x^3+2x^2y-xy^2-2y^3+3y^2+3xy+x+1=0$
3) $2x^3+3x^2y-3xy^2-2y^3+3x^2-3y^2+y=3$
4) $x^3+3xy^2+y^2+2x+y=0$
5) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$
6) $y^2(a^2-x^2)=x^4$
7) $x^2y^3+x^3y^2=x^3+y^3$
8) $xy^3+^3y=a^4$
9) $(x^3+a^3)y=bx^3$
10) Prove that the four asymptotes of the curve $x^2y^2-x^2y-xy^2+x+y+1=0$ form a square.

9.7 Special Case when Asymptotes do not exists.

When we put $mx+c$ at the place of y in the curve and after equating the coefficient of the highest power of x (i.e. x^n) to zero, we get the equation in the variable m say $\phi_n(m)=0$. Solve the equation and find the values of m. If $\phi_n'(m)$ is zero but the second highest coefficient of x (i.e. x^{n-1}) is not zero for one or more values of m, then we can say that the curve has no asymptotes or asymptotes might not exist.

Example 9.13. Find the asymptotes of the curve $y^2 = x$.

Solution. Putting $y = mx + c$ in the equation, we obtain
$(mx + c)^2 = x$
$\implies$ (mx+c)$^2 - x = 0$
Equating the zero, the coefficients of x^2 and x, we get
$m^2 = 0$ and $2mc - 1 = 0$
The first equation gives $m = 0$
If $m = 0$, then the second equation gives $-1 = 0$, which is not possible, hence, $y^2 = x$ has no asymptotes.

9.8 Summary

1. The concept of concavity of the function.

2. A function f has a point of inflection at $x = x_0$ if the curve $y = f(x)$ changes its behaviour from concave upward to concave downward or from concave downward to concave upward, as x passes through x_0

3. A function f is concave up if f' is increasing.

4. A function f is concave down if f' is decreasing.

5. A function f is concave up on I if $f''(x) > 0$ for all $x \in I$.

6. A function f is concave down on I if $f''(x) < 0$ for all $x \in I$.

7. A point x_0 on an open interval I is inflection point if f is concave up on one side and concave down on other side on the point x_0.

8. A straight line at a finite distance from the origin to which a tangent to a curve tends as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve.

9. When x moves to ∞ or $-\infty$, the curve approaches to b, then $y = b$ is called horizontal asymptote.

10. When $x- > a$ from left or right, $y- > \infty$ or $y- > -\infty$,then $x = a$ is called Vertical Asymptote.

11. When $x- > \infty$ or $x- > -\infty$ from left or right, $y- > mx + b$,then $y = mx + b$ is called Oblique Asymptote.

12. Condition when the asymptote does not exist, $\phi'_n(m) = 0$ but of second highest coefficient of x (i.e. x^{n-1}) is not zero for one and more value of m.

9.9 Self Assesement Exercises

Find the asymptotes of the following curves :

1) $f(x) = \sqrt[3]{x^2 + x + 1}$

2) $f(x) = (x^{\frac{2}{3}} - 1)^2$

3) $f(x) = \sin x - \cos x; \quad [-\pi, \pi]$

4) $f(x) = 1 - \tan\left(\frac{x}{2}\right); \quad (-\pi, \pi)$

5) $f(x) = (\sin x + \cos x)^2; \quad [-\pi, \pi]$

6) $4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 = 7$

7) $y^3 - 6xy^2 + 11x^2y - 6x^3 + y + x = 0$

8) $y^3 - xy^2 + 2y^2 + 4y + x = 0$

9) $3x^3 + 2x^2y - 7xy^2 + 2y^3 + 7y^2 - 14xy + 4x + 5y = 0$

10) $x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0$

11) $x^3 + x^2y - xy^2 - y^3 + x^2 - y^2 - 2 = 0$

12) $4x^3 - 3xy^2 - y^3 - xy - y^2 = 1$

13) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$

14) $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0$

15) $2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$

9.10 Solution to In-Text Exercises

Exercise 9.1

1. Concave up : $(-\infty, \infty)$
   Concave down : None
   Inflection Point : None

2. Concave up : $(-\frac{1}{2}, \infty)$
   Concave down : $(-\infty, -\frac{1}{2})$
   Inflection Point : $-\frac{1}{2}$

3. Concave up : $(-\infty, 0), (\frac{2}{3}, \infty)$
   Concave down : $(0, \frac{2}{3})$
   Inflection Point :$0, \frac{2}{3}$

4. Concave Up : $(0, \frac{4-\sqrt{6}}{2})$, $(\frac{4+\sqrt{6}}{2}, \infty)$
Concave Down : $(-\infty, 0)$, $(\frac{4-\sqrt{6}}{2}, \frac{4+\sqrt{6}}{2})$
Inflection Point : $0, \frac{4+-\sqrt{6}}{2}$

Exercise 9.2

1) $x + y = 1, \;\; x - y + 1 = 0, \;\; x + 2y = 0$

2) $x + 2y = 1, \;\; y = x + 1, \;\; x + y = 0$

3) $y = x, \;\; y + 2x + 1 = 0, \;\; x + 2y + 1 = 0$

4) $3x + 1 = 0$

5) $y = + - a, \;\; x = + - b$

6) $x = + - a$

7) $y = + - 1, \;\; x = + - 1$

8) $y = 0, \;\; x = 0$

9) $y = b, \;\; x = -a$

Solution of Self Assessment Exercises

1) Concave Up : $(-2, 1)$
Concave Down : $(-\infty, -2)$, $(1, \infty)$
Inflection Point : $-2, 1$

2) Concave Up : $(-\infty, 0)$, $(0, \infty)$
Concave Down : None
Inflection Point : None

3) Concave Up : $(-\frac{3\pi}{4}, \frac{\pi}{4})$
Concave Down : $(-\pi, -\frac{3\pi}{4})$, $(\frac{\pi}{4}, \pi)$
Inflection Point : $-\frac{3\pi}{4}, \frac{\pi}{4}$

4) Concave Up : $(-\pi, 0)$
Concave Down : $(0, \pi)$
Inflection Point : 0

5) Concave Up : $(-\frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, \pi)$
Concave Down : $(-\pi, \frac{\pi}{2})$, $(0, \frac{\pi}{2})$
Inflection Point : $0, + - \frac{\pi}{2}$

6) $y = -x, \;\; y = x + \frac{2}{3}, \;\; y = 4x + \frac{1}{3}$

7) $y = x, \;\; y = 2x, \;\; y = 3x$

8) $y = 0, \;\; x - y = 1, \;\; x + y + 1 = 0$

9) $y - x + \frac{7}{6}$, $y - 3x + \frac{3}{2} = 0$, $2y + x + \frac{5}{3} = 0$, $106y - 381x + 105 = 0$

10) $y = x$, $y = -x - 1$, $y = -x + 1$

11) $y = x$, $y = -x$, $y = -x - 1$

12) $y = x$, $y = -2x$, $y = -2x - 1$

13) $y = x$, $2y = -x + 1$, $2y = -x - 1$

14) $y = x$, $2y = x$, $2y = x + 1$

15) $x + y = 2$, $x - y + 2 = 0$, $2x - y = 4$

Lesson - 10

Curve Tracing

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Structure

10.1 Learning Objectives

1. To learn the basic criterion for tracing a graph.

2. To trace graph of functions in $x - y$ coordinates.

3. To trace the graphs in polar coordinates.

10.2 Introduction

In previous sections, we have discussed about the concavity of functions and asymptotes.With the help of derivative we can also find that where the function is increasing , decreasing and find the extremum value of function. In this section, we will try to trace the curves with these information's, we will discuss procedures for tracing polynomials, rational functions and polar equations.

10.3 Criterion for Curve Tracing

Following steps (points) are considered for tracing a curve:

1. To obtain the domain of the function.
2. (Symmetries) To find the symmetries of the function. Check that the graph is symmetry about the y-axis or the origin.
3. (x and y intercept) To find the points of intersection of the curve with coordinate axes. Identify all those points where graph intersects x and y-axis.
4. To find the points of maxima or minima of the function using first and second derivative.
5. To check the monotonicity (incressing or decressing) of the function using first derivative.
6. To check the concavity of the curve (function) using first and second derivative..
7. To find the points of inflection, if any.
8. To find the asymptotes of the curve, if exits.(Vertical Asymptote) Identify all the values of x after putting the denominator equal to 0. At each of these values the graph has vertical asymptote. (End Behaviour) Check the behaviour of $f(x)$ when x tends to ∞ and $-\infty$. We get any finite value of $f(x)$ (say l) the $y = l$ is horizaontal asymptote.
9. Check the sign of any function on any open interval by choosing any point from that interval and find the value of tht function on that point.

10.4 Tracing of Polynomial and Rational Functions

Example 10.1. Sketch the graph of $y = x^2 - 1, x \in R$

Solution. Symmetries : Replacing x by $-x$ in the given equation, we have

$$(-x)^2 - 1 = x^2 - 1 = y$$

Since, the equation does not change, so the graph is symmetric about the y-axis.

x **and** y **intercepts :** Substituting $y = 0$ in the given equation, we get

$$x^2 - 1 = 0$$
$$\implies x^2 = 1$$
$$\implies x = -1, 1$$

Therefore the function will intersect x-axis at 1 and -1.
Substituting $x = 0$ in the given equation, we get $y = -1$
Therefore the function will intersect y-axis at -1.

Vertical Asymptotes : There is no fuction of x in the denominator, so there is no vertical asymptote
Sign of y : The sign of y can change at the point $x = -1, 1$. Hence x-axis divides into 3 open intervals.

Table 10.1:

Interval	**Sign of y**
$(-\infty, -1)$	Positive
$(-1, 1)$	Negative
$(1, \infty)$	Positive

End Behaviour : $\lim_{x\to\infty} x^2 - 1 = \infty$
$\lim_{x\to-\infty} x^2 - 1 = \infty$
hence, there is no horizontal asymptote.
Derivative :

$$f(x) = x^2 - 1$$
$$f'(x) = 2x$$
$$f''(x) = 2$$
$$f(x) = 0$$
$$\implies x = -1, 1$$

Table 10.2:

Interval	**Sign of $f'(x)$**	**Increasing/Decreasing**
$(-\infty, -1)$	Positive	Increasing
$(-1, 0)$	Positive	Increasing
$(0, 1)$	Negative	Decreasing
$(1, \infty)$	Negative	Decreasing

$\implies$ The curve is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$
$f'(x) = 0 \implies 2x = 0 \implies x = 0$
$\implies$ The tangent to the curve at the point $x = 0$, is is parallel to $x - axis$.

Table 10.3:

Interval	**Sign of $f''(x)$**	**Concavity**
$(-\infty, -1)$	Positive	Concave Up
$(-1, 0)$	Positive	Concave Up
$(0, 1)$	Positive	Concave Up
$(1, \infty)$	Positive	Concave Up

$\implies$ The curve is concave upward in $(-\infty, \infty)$

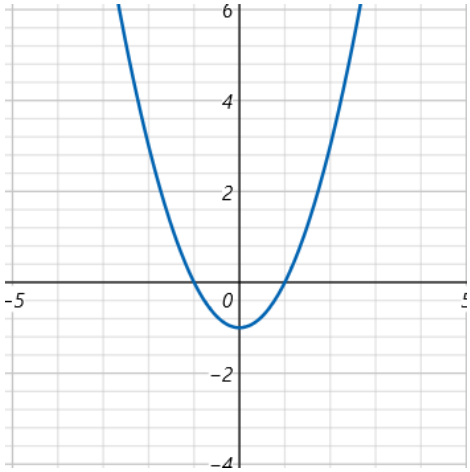


Figure 10.1: $y = x^2 - 1$

Example 10.2. Sketch a graph of the function $y = \frac{2x^2-8}{x^2-16}$

Solution. There is no common factor of numerator and denominator
Symmetries : Replace x by $-x$

$$\frac{2(-x)^2-8}{(-x)^2-16} = \frac{2x^2-8}{x^2-16} = y$$

Equation does not change, so the graph is symmetric about the y-axis.

x **and** y **intercepts :** Substituting $y = 0$ in the given equation,

$$\text{we get} \quad \frac{2x^2-8}{x^2-16} = 0$$
$$\implies 2x^2 - 8 = 0$$
$$\implies 2x^2 = 8$$
$$\implies x = -2, 2$$

$\implies$ The graph of the function intersecting the $x - axis$ at $x = 2$ and $x = -2$
Now, substitute $x = 0$ in the given equation, we get

$$y = \frac{2(0)-8}{0-16} = \frac{1}{2}$$

Therefore the function will intersect y-axis at $\frac{1}{2}$.

Vertical Asymptotes : Put $x^2 - 16 = 0$

$$\implies x^2 = 16$$
$$\implies x = -4, 4$$

Therefore, the function has two vertical asymptotes $x = 4$ and $x = -4$.

Table 10.4:

Interval	Sign of y
$(-\infty, -4)$	Positive
$(-4, -2)$	Negative
$(-2, 2)$	Positive
$(2, 4)$	Negative
$(4, \infty)$	Positive

Sign of y : The y can change its sign at the point $x = -4, -2, 2, 4$. Hence x-axis divides into 5 open intervals.

End Behaviour : $\lim_{x\to\infty} \frac{2x^2-8}{x^2-16} = \lim_{x\to\infty} \frac{2-\frac{8}{x^2}}{1-\frac{16}{x^2}} \implies \frac{2}{1} = 2$

Hence, y=2 is horizontal asymptote.

Derivative : $f(x) = \frac{2x^2-8}{x^2-16}$

Now, $f'(x) = \frac{4x(x^2-16)-2x(2x^2-8)}{(x^2-16)^2}$

$\implies f'(x) = -\frac{48x}{(x^2-16)^2}$

Therefore, $f''(x) = -\frac{48(x^2-16)^2-2(x^2-16)(2x)(48x)}{(x^2-16)^3}$

For $f(x) = 0$

$\implies x = -4, 4$

Table 10.5:

Interval	Sign of $f'(x)$	Increasing/Decreasing
$(-\infty, -4)$	Positive	Increasing
$(-4, 0)$	Positive	Increasing
$(0, 4)$	Negative	Decreasing
$(4, \infty)$	Negative	Decreasing

$f'(x) = 0 \implies \frac{-48x}{(x^2-16)^2} = 0 \implies x = 0$

Table 10.6:

Interval	Sign of $f''(x)$	Concavity
$(-\infty, -4)$	Positive	Concave Up
$(-4, 0)$	Negative	Concave Down
$(0, 4)$	Negative	Concave Down
$(4, \infty)$	Positive	Concave Up

$\implies$ The curve concave upward in $(-\infty, -4) \cup (4, \infty)$ and concave downward in $(-4, 4)$

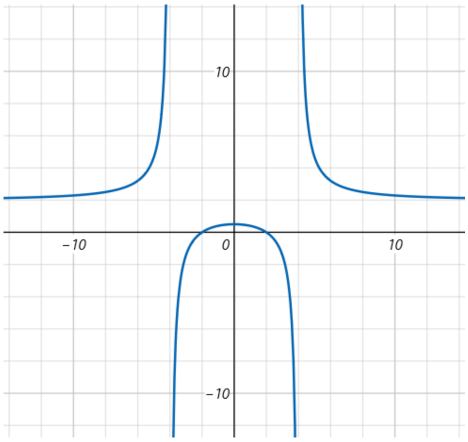


Figure 10.2: $y = \frac{2x^2-8}{x^2-16}$

10.5 Tracing of Functions in the form $y^2 = f(x)$

Example 10.3. Trace the curve $y^2(1+x) = x^2(3-x)$

Solution. Symmetries: Replace y by$-y$

$$(-y^2)(1+x) = y^2(1+x) = x^2(3-x)$$

Equation does not changes, so the graph is symmetric about $x-axis$.

x **and** y **intercepts**: Substituting $y = 0$ in the given equation,we get

$$x^2(3-x) = 0$$
$$\implies \text{x = 0, 3}$$

$\implies$ The graph of the function intersecting the $x-axis$ at $x = 0$ and $x = 3$.
Now, Substituting $x = 0$ in the given equation, we get

$$(-y^2) = 0$$
$$\implies \text{y = 0}$$

Therefore the function will intersect $y-axis$ at 0.

Vertical Asymptotes: Since $y^2 = \frac{(3-x)x^2}{1+x}$
Put $1 + x = 0$

$$1 + x = 0$$
$$\implies \text{x=-1}$$

Therefore, the function has vertical asymptote $x = -1$

Sign of y: The y can change its sign at the point $x = 0, 3$. Hence $x - axis$ divides into 3 open intervals
The given equation may be written as

$$y^2 = \frac{(3-x)x^2}{1+x}$$
$$\implies \text{y=x}\sqrt{\frac{(3-x)}{1+x}} \text{ and } y = -x\sqrt{\frac{(3-x)}{1+x}}$$

Table 10.7:

Interval	**Sign of y**
$(-\infty, 0)$	Positive and Negative both
$(0, 3)$	Positive and Negative both
$(3, \infty)$	Not defined

Derivative: Calculating $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{3-x^2}{\sqrt{3-x}(1+x)^{\frac{3}{2}}}$$
$$\frac{dy}{dx} = 0$$
$$\implies \text{3-x}^2\frac{}{\sqrt{3-x}(1+x)^{\frac{3}{2}}=0}$$
$$\text{Now, } 3 - x^2 = 0$$
$$\text{and } x = \sqrt{3}$$

Table 10.8:

Interval	**Sign of $f'(x)$**	**Increasing/Decreasing**
$(-\infty, \sqrt{3})$	Positive	Increasing
$(\sqrt{3}, \infty)$	Negative	Decreasing

Since the curve is symmetrical about $x - axis$, the second component is the reflection of the curve in the $x - axis$. hence, given curve is plotted as given in figure 10.3.

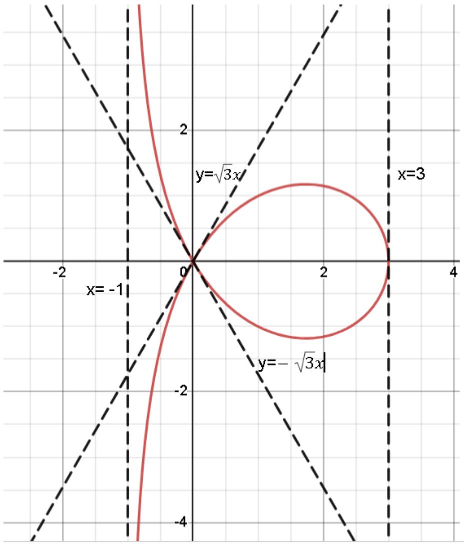


Figure 10.3:

In-text Exercise 10.1

Sketch the graph of the functions

1. $f(x) = x^6 - 3x^4 + 2x^2$
2. $f(x) = x^3 - 5x^2 - x + 5$
3. $(x-2)^2(2x+3)$
4. $x^3 + 4x^2 + x - 6$
5. $(x+3)(x-2)^2(x+1)^3$
6. $y^2(1+x) = x(5+x)$

10.6 Tracing of Curves in Polar Forms

We know the relationship between polar and rectangular coordinates. If the polar coordinates of a point are (r, θ) and its rectangular coordinates are (x, y) then we have $x = r\cos\theta$, and $y = \sin\theta$
$\implies x^2 + y^2 = r^2$
and $\tan\theta = \frac{y}{x}$

Example 10.4. Convert rectangular coordinate into polar coordinate for the point $(r, \theta) = (6, \frac{2\pi}{3})$

Solution. We have $r = 6, \qquad \theta = \frac{2\pi}{3}$
Therefore, $x = r\cos\theta = 6\cos\frac{2\pi}{3} = 6\frac{-1}{2} = -3$
and $y = r\sin\theta = 6\sin\frac{2\pi}{3} = 6\frac{\sqrt{3}}{2} = 3\sqrt{3}$
Hence, the rectangular Coordinates are $(-3, 3\sqrt{3})$

Example 10.5. Convert polar coordinate into rectangular coordinate for the point $(-2, -2\sqrt{3})$

Solution. We have $x = -2, \qquad y = -2\sqrt{3}$
and $r^2 = x^2 + y^2 \implies r^2 = (-2)^2 + (-2\sqrt{3})^2$
$\implies r^2 = 4 + 12 = 16$
$\implies r = 4$

Also, $\tan\theta = \frac{y}{x} = \frac{-2\sqrt{3}}{-2} = \sqrt{3}$
$\implies \theta = \frac{4\pi}{3}$
Therefore the polar coordinates are $(4, \frac{4\pi}{3})$

Example 10.6. Trace the curve $r = 1$.

Solution. For the given curve the value of r is always 1 with respect to any value of θ. Therefore trace the circle at the unit distance from the origin as given in figure 10.4.

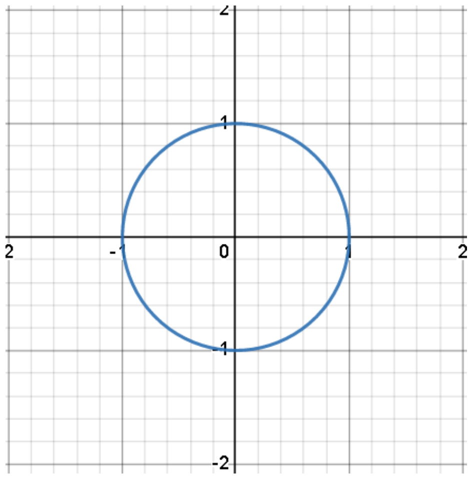


Figure 10.4: $r = 1$

Example 10.7. Trace the curve $\theta = \frac{\pi}{4}$.

Solution. For the given curve the value of θ is always $\frac{\pi}{4}$ with respect to any value of r. Therefore trace the line at the angle $\frac{\pi}{4}$ from the origin as given in figure 10.5.

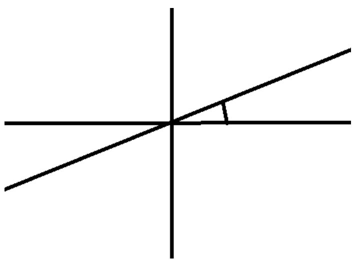


Figure 10.5: $\theta = \frac{\pi}{4}$

Example 10.8. Trace the curve $r = \sin\theta$.

Solution. Table 10.9 shows the coordinates of points of $r = \sin\theta$.

Table 10.9:

θ	$r = \sin\theta$	(r, θ)
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$(\frac{1}{2}, \frac{\pi}{6})$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$(\frac{\sqrt{3}}{2}, \frac{\pi}{3})$
$\frac{\pi}{2}$	1	$(1, \frac{\pi}{2})$
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$(\frac{\sqrt{3}}{2}, \frac{2\pi}{3})$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$(\frac{1}{2}, \frac{5\pi}{6})$
π	0	$(0, \pi)$

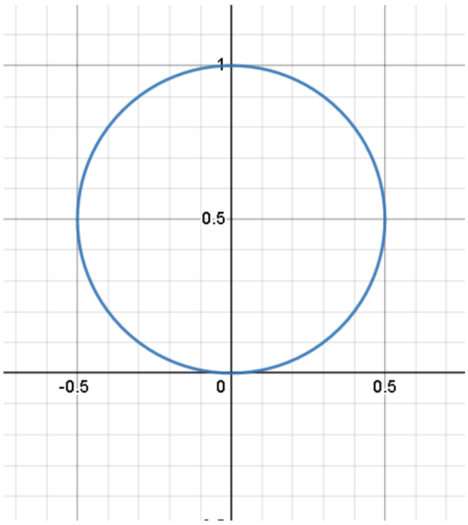


Figure 10.6: $r = \sin\theta$

In-text Exercise 10.2

Find the rectangular coordinate for the given polar coordinates.

1. $(6, \frac{\pi}{6})$
2. $(7, \frac{2\pi}{3})$
3. $(-6, -\frac{5\pi}{6})$

4. $(0, -\pi)$

Find the polar coordinate for the given rectangular coordinates

5. $(2\sqrt{3}, -2)$
6. $(0, -2)$
7. $(-8, -8)$
8. $(-3, \sqrt{3})$
9. $(1, 1)$

10.7 Summary

1. Sketch the graph of Rational or Polynomial function $f(x) = \frac{P(x)}{Q(x)}$ (for Polynomial $Q(x) = 1$) if there is no common factor of $P(x)$ and $Q(x)$.

   - **Step 1 :** (Symmetries) Check that the graph is symmetry about the y-axis or the origin.

   - **Step 2 :** (x and y intercepts) Identify all those points where graph intersects x and y-axis.
     Step 3 : (Vertical Asymptotes) Identify all the values of x after putting $Q(x) = 0$.At each of these values the graph has vertical asymptope.

   - **Step 4 :** (Sign of $f(x)$) Locate all those points at which $f(x)$ can change signs. These points are at the vertical asymptotes or where graph intersect at $x - axis$. Find the interval where $f(x)$ is positive or negative with the help of these points.

   - **Step 5 :** (End Behaviour) Check the behaviour of $f(x)$ when x tends to ∞ and $-\infty$. We get any finite value of $f(x)$ (say l) then $y = l$ is horizontal asymptote.

   - **Step 6 :** (Derivative) Find first and second derivative of function i.e. $f'(x)$ and $f''(x)$.

   - **Step 7 :** (Conclusion and Graph) When we find derivative of $f(x)$ then we conclude that where the graph of increasing or decreasing with the help of derivative. After computing the second derivative, we can conclude that where the graph is concave up and concave down with the help of first and second derivative test.
     .

2. Convert Polar coordinates to Rectangular Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

3. Convert Rectangular coordinates to Polar Coordinates

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{x}{y}$$

10.8 Self Assessment Exercises

Sketch the graph of rational functions.

1. $f(x) = \frac{2x-6}{4-x}$
2. $f(x) = \frac{x}{x^2-4}$
3. $f(x) = \frac{x^2}{x^2+4}$
4. $f(x) = \frac{x^3+1}{x^3-1}$
5. $f(x) = \frac{(3x+1)^2}{(x-1)^2}$
6. Find the rectangular coordinate of $(7, \frac{7\pi}{6})$
7. Find the polar coordinate of $(-5, 0)$ Sketch the graph of given curves in polar form
8. $\theta = \frac{\pi}{3}$
9. $r = 3$
10. $r = 6\sin\theta$
11. $r = 3(1 + \sin\theta$
12. $r = 4 - 4\cos\theta$
13. $r = -1 - \cos\theta$
14. $r = 3 - \sin\theta$
15. $r = 5 + 3\sin\theta$
16. $r = -3 - 4\sin\theta$
17. $r^2 = 16\sin 2\theta$
18. $r = 4\theta \qquad (\theta \leq 0)$
19. $r = -2\cos 2\theta$
20. $r = 9\sin 4\theta$

10.9 Solution to In-Text Exercises

Exercise 10.1

1.

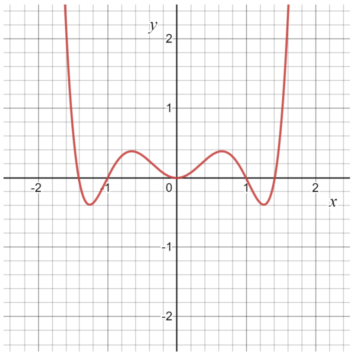


Figure 10.7: $f(x) = x^6 - 3x^4 + 2x^2$

2.

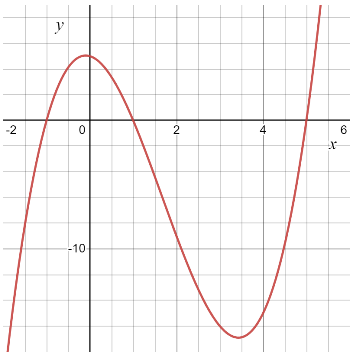


Figure 10.8: $f(x) = x^3 - 5x^2 - x + 5$

3.

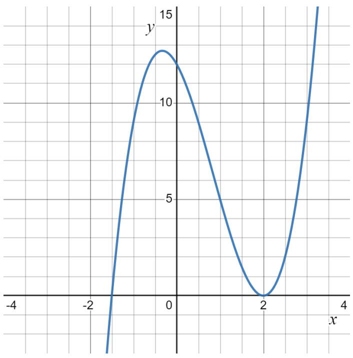


Figure 10.9: $(x-2)^2(2x+3)$

4.

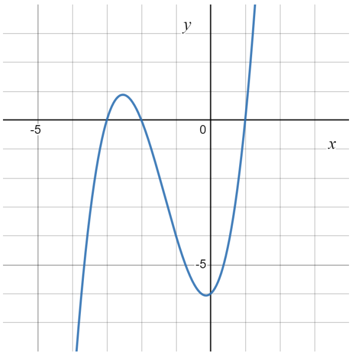


Figure 10.10: $x^3 + 4x^2 + x - 6$

5.

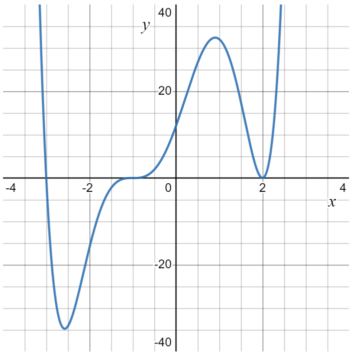


Figure 10.11: $(x + 3)(x - 2)^2(x + 1)^3$

6.

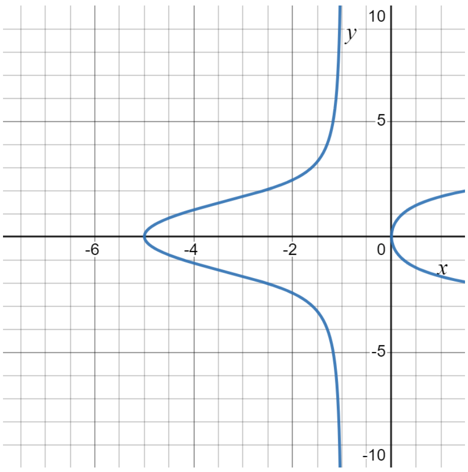


Figure 10.12: $y^2(1 + x) = x(5 + x)$

Exercise 10.2

1. $(3\sqrt(3), 3)$

2. $(\frac{-7}{2}, \frac{7\sqrt{3}}{2})$

3. $(3\sqrt(3), 3)$

4. $(0, 0)$

5. $(4, \frac{11\pi}{6}), (4, \frac{-\pi}{6})$

6. $(2, \frac{3\pi}{3}), (2, \frac{-\pi}{2})$

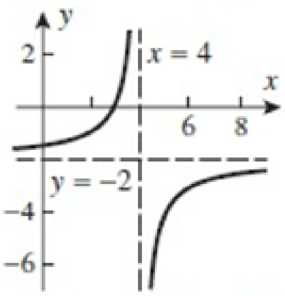
7. $(8\sqrt{2}, \frac{5\pi}{4}), (8\sqrt{2}, \frac{-3\pi}{4})$

8. $(6, \frac{2\pi}{3}), (6, \frac{-4\pi}{3})$

9. $(\sqrt{2}, \frac{\pi}{4}), (\sqrt{2}, \frac{-7\pi}{4})$

Solution of Self Assessment Exercise

1.



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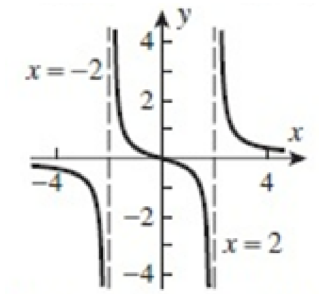

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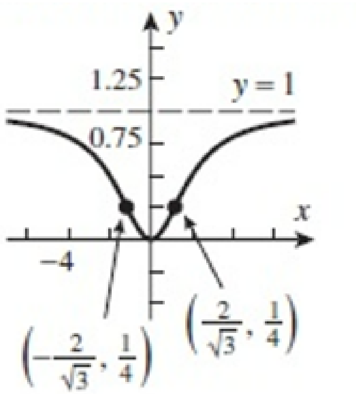

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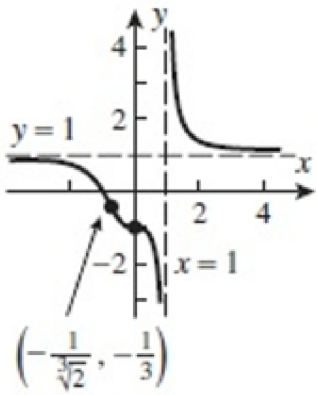

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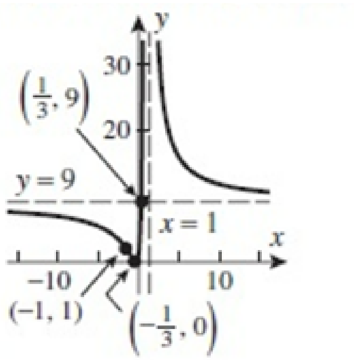
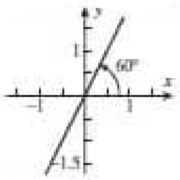


6. $(\frac{-7\sqrt{3}}{2}, \frac{7}{2})$
7. $(5, \pi)$

8.



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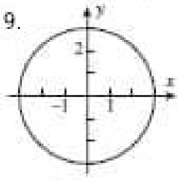

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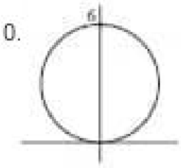
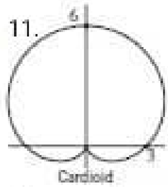


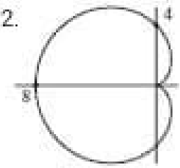
Circle

11.



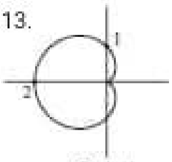
Cardioid

12.



Cardioid

13.



Cardioid

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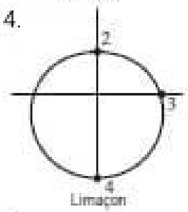
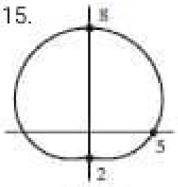


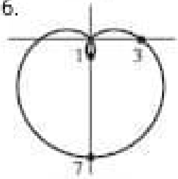
Limaçon

15.



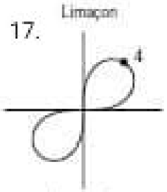
Limaçon

16.



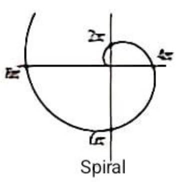
Limaçon

17.



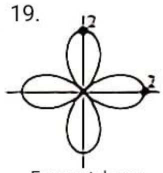
Lemniscate

18.



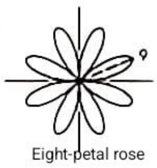
Spiral

19.



Four-petal rose

20.



Eight-petal rose

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